Interactia radiatiei laser cu substanta
L5

1. Classical electromagnetic field
   - The monochromatic plane wave
   - The electromagnetic potentials
   - Stokes parameters
   - The multimode field
2. Quantized electromagnetic field
   - The field operators
   - The Fock space
   - Heisenberg picture
   - Coherent states
   - Phase operator
   - Semiclassical approximation
3. Motion of the classical particle in electromagnetic fields
   - NR dipole approximation
   - Non-relativistic non-dipole approximation

- The diagramatic method for monochromatic case
- Renormalization of perturbation theory

The monochromatic plane wave I

Maxwell equations for the free electromagnetic field:

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \boldsymbol{\nabla} \times \mathbf{B}\,, \qquad \frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \mathbf{E}\,,$$
$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0\,, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0\,. \tag{1}$$

or

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \Delta \mathbf{E}\,, \qquad \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = \Delta \mathbf{B}\,. \tag{2}$$

monochromatic plane wave:

$$\mathbf{E}(\mathbf{r}, t) = E_{0x'} \sin(\omega t - \boldsymbol{\kappa} \cdot \mathbf{r} + \delta_1)\,\mathbf{e}_1 + E_{0y'} \sin(\omega t - \boldsymbol{\kappa} \cdot \mathbf{r} + \delta_2)\,\mathbf{e}_2 \ . \tag{3}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\boldsymbol{\kappa} \times \mathbf{E}(\mathbf{r}, t)}{\omega}\,, \tag{4}$$

and

- $\mathbf{e}_1$, $\mathbf{e}_2$, $\hat{\boldsymbol{\kappa}}$ form a right-handed coordinate system.

The monochromatic plane wave II

- $\omega = c|\boldsymbol{\kappa}|$

$$E_{x'} = E_{0x'} \sin(\omega t - \boldsymbol{\kappa}\cdot\mathbf{r} + \delta_1)\,, \qquad E_{y'} = E_{0y'} \sin(\omega t - \boldsymbol{\kappa}\cdot\mathbf{r} + \delta_2)\,. \tag{5}$$

$$\frac{E_{x'}^2}{E_{0x'}^2} + \frac{E_{y'}^2}{E_{0y'}^2} - 2\frac{E_{x'}E_{y'}}{E_{0x'}E_{0y'}}\cos\delta = \sin^2\delta\,. \tag{6}$$

i.e. equation of an ellipse (with $\delta = \delta_1 - \delta_2$); for $\delta = \frac{\pi}{2}$ (mod π) the ellipse semiaxes are alonf $\mathbf{e}_1$, $\mathbf{e}_2$.

If $\mathbf{e}_1$ is along the major semiaxis, ($\mathbf{e}_1 \to \mathbf{s}_1$, $\mathbf{e}_2 \to \mathbf{s}_2$)

$$\mathbf{E}(\mathbf{r},t) = E_{01}\sin(\omega t - \boldsymbol{\kappa}\cdot\mathbf{r} + \delta_0)\,\mathbf{s}_1 - E_{02}\cos(\omega t - \boldsymbol{\kappa}\cdot\mathbf{r} + \delta_0)\,\mathbf{s}_2\ . \tag{7}$$

The general case

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathcal{E}_0\,\mathbf{s}\,e^{i(\boldsymbol{\kappa}\cdot\mathbf{r}-\omega t)}\right], \tag{8}$$

with $\mathcal{E}_0$ defined as

$$\mathcal{E}_0 = i\sqrt{E_{0x'}^2 + E_{0y'}^2}\,e^{-i\delta_1}\,, \tag{9}$$

The monochromatic plane wave III

and the *complex* polarization vector

$$\mathbf{s} = \cos\zeta/2\,\mathbf{e}_1 + e^{-i\delta}\,\sin\zeta/2\,\mathbf{e}_2\,, \tag{10}$$

with

$$\cos\zeta/2 \;=\; \frac{E_{0x'}}{\sqrt{E_{0x'}^2+E_{0y'}^2}}, \qquad \sin\zeta/2 \;=\; \frac{E_{0y'}}{\sqrt{E_{0x'}^2+E_{0y'}^2}}\,. \tag{11}$$

$$\mathbf{s}\cdot\boldsymbol{\kappa} = 0, \qquad \mathbf{s}\cdot\mathbf{s}^{\star} \equiv |\,\mathbf{s}\,|^2 = 1\,. \tag{12}$$

- linear polarization:

$$\delta = 0, \pm\pi \longrightarrow \mathbf{s} = \cos\zeta/2\,\mathbf{e}_1 \pm \sin\zeta/2\,\mathbf{e}_2 = \mathbf{s}^*\,, \tag{13}$$

- circular polarization

$$\delta = \pm\pi/2, \quad E_{0y'} = E_{0x'} \quad \longrightarrow \mathbf{s} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \mp i\mathbf{e}_2)\,. \tag{14}$$

The monochromatic plane wave IV

if $\mathbf{e}_1 = \mathbf{s}_1$, $\mathbf{e}_2 = \mathbf{s}_2$ (i.e. ellipse principal axis along the coordinate frame unit vectors)

$$\mathbf{s} = \cos\zeta/2\,\mathbf{s}_1 + i\,\sin\zeta/2\,\mathbf{s}_2\,, \tag{15}$$

and the electric field becomes

$$\mathbf{E} = E_0\left[\,\cos\zeta/2\sin(\omega t - \boldsymbol{\kappa}\cdot\mathbf{r} + \delta_0)\,\mathbf{s}_1 - \sin\zeta/2\cos(\omega t - \boldsymbol{\kappa}\cdot\mathbf{r} + \delta_0)\,\mathbf{s}_2\,\right]\,, \tag{16}$$

with

$$E_0 \equiv \sqrt{E_{01}^2 + E_{02}^2}\,. \tag{17}$$

NB:

$$E_{01}^2 + E_{02}^2 = E_{0x'}^2 + E_{0y'}^2\,, \tag{18}$$

The energy density

$$w(\mathbf{r},t) = \frac{\epsilon_0}{2}\left(E^2(\mathbf{r},t) + c^2 B^2(\mathbf{r},t)\right) \ . \tag{19}$$

For a plane wave

$$w(\mathbf{r},t) = \epsilon_0\, E^2(\mathbf{r},t)\,, \tag{20}$$

$$\rho \equiv \frac{1}{T}\int_0^T w(\mathbf{r},t)dt = \frac{\epsilon_0}{2}\,(E^2_{0x'} + E^2_{0y'}) = \frac{\epsilon_0}{2}\,E_0^2\,. \tag{21}$$

The electromagnetic field intensity

$$I = \rho\, c = \frac{\epsilon_0}{2}\, c\, E_0^2 \ = \frac{\epsilon_0}{2}\, c \mid \mathcal{E}_0 \mid^2 \ . \tag{22}$$

The total energy

$$\mathcal{W}_{\mathrm{V}} \ = \frac{\epsilon_0}{2}\int_{\mathrm{V}}\left(\mathbf{E}^2 \ + c^2\mathbf{B}^2\ \right)\ dv = \rho V \tag{23}$$

The electromagnetic potentials

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t}\,, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}\,. \tag{24}$$

gauge invariance:

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla} f\,, \quad \Phi' = \Phi - \frac{\partial f}{\partial t}\,, \tag{25}$$

Examples:

- Coulomb gauge: $\Phi = 0$, iar $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$.

$$\mathbf{A}(\boldsymbol{\kappa};\mathbf{r},t) = \frac{1}{\omega}\left[E_{0x'}\,\cos(\omega t - \boldsymbol{\kappa}\cdot\mathbf{r} + \delta_1)\,\mathbf{e}_1 + E_{0y'}\,\cos(\omega t - \boldsymbol{\kappa}\cdot\mathbf{r} + \delta_2)\,\mathbf{e}_2\right]. \tag{26}$$

- Coulomb gauge + "good" choice of the coordinte system:

$$\mathbf{A}(\mathbf{r},t) = A_0\left[\cos\frac{\zeta}{2}\cos(\omega t - \boldsymbol{\kappa}\cdot\mathbf{r} + \delta_0)\,\mathbf{s}_1 + \sin\frac{\zeta}{2}\sin(\omega t - \boldsymbol{\kappa}\cdot\mathbf{r} + \delta_0)\,\mathbf{s}_2\right],\; A_0 = \frac{E_0}{\omega} \tag{27}$$

Stokes parameters

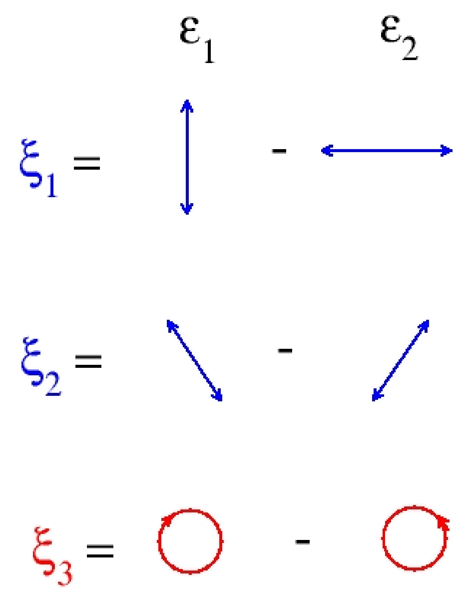
Definition in classical electrodynamics

$$\mathbf{E}(t,\mathbf{r}) = E_0 Re\left\{\varepsilon e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}\right\}$$

One defines two unit vectors $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ such that

$$\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_2 = 0, \quad \boldsymbol{\epsilon}_1 \cdot \mathbf{k} = 0, \quad \boldsymbol{\epsilon}_2 \cdot \mathbf{k} = 0$$

$$\varepsilon_i = \boldsymbol{\varepsilon} \cdot \boldsymbol{\epsilon}_i, \quad i = 1, 2$$



- $\xi_1 \sim \mathcal{I}_{\boldsymbol{\epsilon}_1} - \mathcal{I}_{\boldsymbol{\epsilon}_2}$,
- $\xi_2 \sim \mathcal{I}_{(\boldsymbol{\epsilon}_1+\boldsymbol{\epsilon}_2)/\sqrt{2}} - \mathcal{I}_{(\boldsymbol{\epsilon}_1-\boldsymbol{\epsilon}_2)/\sqrt{2}}$
- $\xi_3 \sim \mathcal{I}_{(\boldsymbol{\epsilon}_1+I\boldsymbol{\epsilon}_2)/\sqrt{2}} - \mathcal{I}_{(\boldsymbol{\epsilon}_1-I\boldsymbol{\epsilon}_2)/\sqrt{2}}$
  where $\mathcal{I}_\epsilon$ intensity corresponding to the component of **E** along ϵ.

The multimode field I

$$\mathbf{A}(\mathbf{r},t) = \int_{\boldsymbol{\kappa}} \sum_{\mathbf{s}(\boldsymbol{\kappa})} \mathbf{A}(\boldsymbol{\kappa}, \mathbf{s}(\boldsymbol{\kappa}); \mathbf{r}, t)\, d\boldsymbol{\kappa}\,. \quad (28)$$

Periodic conditions in the volume $V = L^3$

$$\mathbf{A}(x+L, y, z, t) \;=\; \mathbf{A}(x, y+L, z, t) \;=\; \mathbf{A}(x, y, z+L, t) \;=\; \mathbf{A}(x, y, z, t)\,. \quad (29)$$

$$e^{i\kappa_x L} \;=\; e^{i\kappa_y L} \;=\; e^{i\kappa_z L} \;=\; 1\,,$$

$$\kappa_x = \frac{2\pi}{L} n_x\,, \quad \kappa_y = \frac{2\pi}{L} n_y\,, \quad \kappa_z = \frac{2\pi}{L} n_z\,, \quad (30)$$

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathrm{mod}} \mathbf{A}_{\mathrm{mod}}(\mathbf{r},t) \quad \mathrm{cu} \quad \mathbf{A}_{\mathrm{mod}}(\mathbf{r},t) = \mathrm{Re}[\mathcal{A}_{0,\,\mathsf{mod}} \mathbf{s}(\boldsymbol{\kappa}) e^{i(\boldsymbol{\kappa}\cdot\mathbf{r} - \omega t)}]\,, \quad (31)$$

$$\mathbf{A}(\mathbf{r},t) = \mathrm{Re}\{\sum_{\boldsymbol{\kappa}} \sum_{\lambda=a,b} \mathbf{s}_\lambda(\boldsymbol{\kappa}) C_\lambda(\boldsymbol{\kappa}, t) \frac{e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}}{\sqrt{V}}\}\,. \quad (32)$$

The multimode field II

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\{\, i \sum_{\boldsymbol{\kappa}} \sum_{\lambda=a,b} \omega(\kappa)\, \mathbf{s}_\lambda(\boldsymbol{\kappa})\, C_\lambda(\boldsymbol{\kappa},t)\, \frac{e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}}{\sqrt{V}} \} \tag{33}$$

$$\mathbf{B}(\mathbf{r},t) = \mathrm{Re}\{\, i \sum_{\boldsymbol{\kappa}} \sum_{\lambda=a,b} \boldsymbol{\kappa} \times \mathbf{s}_\lambda(\boldsymbol{\kappa})\, C_\lambda(\boldsymbol{\kappa},t)\, \frac{e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}}{\sqrt{V}} \}\,. \tag{34}$$

$$\mathcal{W}_{\mathrm{V}} = \frac{\epsilon_0}{2} \int_{\mathrm{V}} \left(\mathbf{E}^2 + c^2 \mathbf{B}^2 \right) dv\,, \tag{35}$$

$$\mathcal{W}_{\mathrm{V}} = \frac{\epsilon_0}{2} \sum_{\boldsymbol{\kappa}} \sum_{\lambda=a,b} \omega^2(\kappa)\ \left| C_\lambda(\boldsymbol{\kappa},t) \right|^2 \ . \tag{36}$$

Dipole approximation: homogeneous fields

$$\mathbf{E}(\mathbf{r},t) \to \mathbf{E}(t), \quad \mathbf{B}(\mathbf{r},t) \to 0 \tag{37}$$

The field operators in the Schrödinger picture I

Schrodinger picture: (operators are time independent)

$$\mathbf{A}^{\rm op}_{\rm mod}(\mathbf{r}) = \sqrt{\frac{\hbar}{2\,\epsilon_0\,\omega\,V}}\,[\,\mathbf{s}\,e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}\,\hat{a} \;+\; \mathbf{s}^*\,e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}}\,\hat{a}^\dagger\,]\,, \tag{38}$$

$$[\,\hat{a}, \hat{a}^\dagger] = \hat{I}\;. \tag{39}$$

$$\mathbf{E}^{\rm op}_{\rm mod}(\mathbf{r}) = \sqrt{\frac{\hbar\,\omega}{2\,\epsilon_0\,V}}\;i\,[\,\mathbf{s}\,e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}\,\hat{a} \;-\; \mathbf{s}^*\,e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}}\,\hat{a}^\dagger\,]\,, \tag{40}$$

$$\mathbf{B}^{\rm op}_{\rm mod}(\mathbf{r}) = i\,\sqrt{\frac{\hbar}{2\,\epsilon_0\,V\,\omega}}\boldsymbol{\kappa}\times[\,\mathbf{s}\,e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}\,\hat{a} \;-\; \mathbf{s}^*\;e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}}\,\hat{a}^\dagger\,]\,. \tag{41}$$

$$\mathbf{A}^{\rm op}(\mathbf{r}) = \sum_{\rm mod}\mathbf{A}^{\rm op}_{\rm mod}(\mathbf{r}). \tag{42}$$

independent modes

$$[\,\hat{a}_{{\rm mod}_1}, \hat{a}_{{\rm mod}_2}] \;=\; [\,\hat{a}_{{\rm mod}_1}, \hat{a}^\dagger_{{\rm mod}_2}] \;=\; 0,$$

The field operators in the Schrödinger picture II

$$\left[\left(E^{\mathrm{op}}_{\mathrm{mod}}\right)_j(\mathbf{r}_1),\left(E^{\mathrm{op}}_{\mathrm{mod}}\right)_k(\mathbf{r}_2)\right] = -\frac{i\hbar\omega}{\epsilon_0 V} s_j s_k \sin\left(\boldsymbol{\kappa}\cdot(\mathbf{r}_1-\mathbf{r}_2)\right)\ \hat{I}\,,$$
$$\left[\left(E^{\mathrm{op}}_{\mathrm{mod}}\right)_j(\mathbf{r}_1),\left(B^{\mathrm{op}}_{\mathrm{mod}}\right)_k(\mathbf{r}_2)\right] = \frac{i\hbar}{\epsilon_0 V} s_j(\boldsymbol{\kappa}\times\mathbf{s})_k \cos\left(\boldsymbol{\kappa}\cdot(\mathbf{r}_1-\mathbf{r}_2)\right)\ \hat{I}\,, \tag{43}$$

The Fock space I

$$\hat{H}_{\rm rad} = \sum_{\rm mod} \hat{H}^{\rm op}_{\rm mod} = \sum_{\rm mod} \hbar\omega_{\rm mod}\hat{a}^{\dagger}_{\rm mod}\hat{a}_{\rm mod} \ . \tag{44}$$

$$H^{\rm op}_{\rm mod} \ = \ \hat{a}^{\dagger}\hat{a}\ \hbar\omega \ . \tag{45}$$

$$\hat{H}^{\rm op}_{\rm mod} \mid n; {\rm mod}\rangle \ = \ n\hbar\omega \mid n; {\rm mod}\rangle \ , \qquad n \geq 0 \ . \tag{46}$$

$$\langle n; {\rm mod} \mid n'; {\rm mod}\rangle = \delta_{nn'} \ . \tag{47}$$

$$\hat{N}_{\rm mod} \equiv \hat{a}^{\dagger}_{\rm mod}\,\hat{a}_{\rm mod} \qquad \mid n\rangle \ = \ \frac{\left(\hat{a}^{\dagger}\right)^{n}}{\sqrt{n!}} \mid 0\rangle \ . \tag{48}$$

$$\begin{aligned} \hat{a} \mid n\rangle \ &= \ \sqrt{n} \mid n-1\rangle \ , \ \ (n \neq 0), \qquad \hat{a} \mid 0\rangle = \mid {\rm zero}\rangle \ , \\ \hat{a}^{\dagger} \mid n\rangle \ &= \ \sqrt{n+1} \mid n+1\rangle \ . \end{aligned} \tag{49}$$

$$\mid {\rm Fock}\rangle \ \equiv \ \prod_{\rm mod} \otimes \mid n; {\rm mod}\rangle = \mid n_1, n_2, ..., n_p, ...\rangle \tag{50}$$

The Fock space II

The total energy

$$W_{\mathrm{Fock}} = \sum_{\mathrm{mod}} n_{\mathrm{mod}} \hbar\omega_{\mathrm{mod}} \ . \tag{51}$$

Stationary states (Schrödinger picture)

$$|\,\Psi^{\mathrm{Fock}}_{\mathrm{stat}}\rangle \ = |\,\mathrm{Fock}\rangle e^{-\frac{i}{\hbar} W_{\mathrm{Fock}}\ t} \ . \tag{52}$$

Energy density

$$\rho_V = \frac{1}{V} \sum_{\mathrm{mod}} n_{\mathrm{mod}} \hbar\omega_{\mathrm{mod}} \ .$$

$$\frac{1}{V} \sum_{\mathrm{mod}} \ldots \to \frac{1}{(2\pi)^3} \int_{\boldsymbol{\kappa}} \sum_{\lambda=1,2} \ldots d\,\boldsymbol{\kappa} \ .$$

One uses spherical coordinates of $\boldsymbol{\kappa}$

$$\rho = \frac{1}{(2\pi)^3} \int d\Omega_{\hat{\boldsymbol{\kappa}}} \sum_{\lambda=1,2} \int_0^\infty \kappa^2 n(\boldsymbol{\kappa}, \mathbf{s}_\lambda)\ \hbar\ \omega\ d\ \kappa\ ,$$

The Fock space

expectation values

$$\begin{aligned}\overline{\mathbf{E}}(\mathbf{r}) &\equiv <\Psi_{\mathrm{stat}}^{\mathrm{Fock}} \mid \mathbf{E}^{\mathrm{op}} \mid \Psi_{\mathrm{stat}}^{\mathrm{Fock}}>=<\mathrm{Fock} \mid \mathbf{E}^{\mathrm{op}} \mid \mathrm{Fock}>=0\;, \\ \overline{\mathbf{B}}(\mathbf{r}) &\equiv <\Psi_{\mathrm{stat}}^{\mathrm{Fock}} \mid \mathbf{B}^{\mathrm{op}} \mid \Psi_{\mathrm{stat}}^{\mathrm{Fock}}>=0\;. \end{aligned} \tag{53}$$

$$\Delta E \equiv \sqrt{<\Psi \mid \mathbf{E}^{\mathrm{op}} \cdot \mathbf{E}^{\mathrm{op}} \mid \Psi> - <\Psi \mid \mathbf{E}^{\mathrm{op}} \mid \Psi> \cdot <\Psi \mid \mathbf{E}^{\mathrm{op}} \mid \Psi>}\;. \tag{54}$$

for one mode

$$\Delta E_{\mathrm{mod}} = \sqrt{\frac{\hbar\omega}{\epsilon_0 V}}(n + 1/2)\;. \tag{55}$$

Heisenberg picture

$$\hat{U}_{\mathrm{S}\rightarrow\mathrm{H}}(t) = e^{\frac{i}{\hbar}\hat{H}_{\mathrm{rad}}\ t}.$$

Time dependent operators and time independent states;

$$|\ \Psi^{\mathrm{H}}_{\mathrm{stat}\ ,\ \mathrm{Fock}} >=|\ \mathrm{Fock} >\ , \tag{56}$$

$$\hat{a}^{H}(t) \ = \ \hat{U}_{\mathrm{S}\rightarrow\mathrm{H}}\,\hat{a}\,\hat{U}^{\dagger}_{\mathrm{S}\rightarrow\mathrm{H}} = \hat{a}\,e^{-i\omega t}, \tag{57}$$

$$\mathbf{A}^{\mathrm{op,H}}(\mathbf{r},t) = \sum_{\mathrm{mod}}\sqrt{\frac{\hbar}{2\epsilon_0 V\omega}}[\,\mathbf{s}\ e^{i(\boldsymbol{\kappa}\cdot\mathbf{r}-\omega t)}\ \hat{a}_{\mathrm{mod}}\ +\ \mathbf{s}^{*}\ e^{-i(\boldsymbol{\kappa}\cdot\mathbf{r}-\omega t)}\ \hat{a}^{\dagger}_{\mathrm{mod}}\ ]\,. \tag{58}$$

Coherent states I

Consider the one-mode case and Schrodinger picture; a coherent state: eigenstate of the annihilation operator.

$$\hat{a} \mid z\rangle \;=\; z \mid z\rangle \; . \tag{59}$$

The set of Fock states is a basis, i.e.

$$\mid z\rangle = \sum_{n=0}^{\infty} c_n \mid n\rangle \; . \tag{60}$$

$$\sqrt{n+1}\, c_{n+1} = z\, c_n \; , \quad c_n = \frac{z^n}{\sqrt{n!}}\, c_0 . \tag{61}$$

Normalization

$$\langle z \mid z\rangle = 1 \; , \tag{62}$$

$$\mid c_0 \mid^2 \sum_{n=0}^{\infty} \frac{\mid z \mid^2}{n!} = \mid c_0 \mid^2 \; e^{|z|^2} \; = \; 1 \; , \tag{63}$$

Coherent states II

i.e. $\mid c_0 \mid^2 = \ e^{-|z|^2}$. Up to an arbitrary phase

$$\mid z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \mid n\rangle \ , \tag{64}$$

NB:

- $|z\rangle$ is not eigenvector of N for arbitrary z
- $|z=0\rangle$ is the vacuum state (and eigenvector of N)

Propertis of coherent states:

$$\langle z_1 \mid z_2\rangle \ = \ \exp\left[-\frac{1}{2} \mid z_1 - z_2 \mid^2 + i\,\mathrm{Im}(z_1^* z_2)\right] \ . \tag{65}$$

i.e. $\langle z_1 \mid z_2\rangle$ are not orthogonal if $z_1 \neq z_2$ but they are *almost* orthogonal if $|z_1 - z_2|$ is large.

$$\langle z \mid \hat{a}^\dagger \mid z\rangle \ = \ z^* \ . \tag{66}$$

Coherent states III

Consider an electromagnetic field in the state

$$\mid \Psi(t_0)\rangle = \mid z\rangle \otimes \mid \chi\rangle \ , \tag{67}$$

i.e. a coherent state in one mode and arbitrary state $id\chi\rangle$ for the other modes. The probability to find n photons in the mode coresponding to $\mid z\rangle$ is

$$p(n) \ = \mid c_n \mid^2 = e^{-|z|^2} \frac{\mid z \mid^{2n}}{n!} \ . \tag{68}$$

i.e. Poisson distribution.

$$\overline{n} = \mid z \mid^2 \ , \tag{69}$$

$$\overline{n^2} = \mid z \mid^4 + \mid z \mid^2, \qquad \overline{(\delta n)^2} = \overline{n} = \mid z \mid^2 \ . \tag{70}$$

$$\frac{\Delta \ n}{\overline{n}} = \frac{1}{\mid z \mid} = \frac{1}{\sqrt{\overline{n}}} \ , \tag{71}$$

small if $\overline{n}$ is large

Coherent states IV

Time evolution of a coherent state:

$$| \Psi(t_0)\rangle = \sum_{n=0}^{\infty} c_n \mid n\rangle \;\longrightarrow | \Psi(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-\frac{i}{\hbar}E_n(t-t_0)} \mid n\rangle \; .$$

our case:

$$| \Psi(t)\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{\left(z e^{-i\omega(t-t_0)}\right)^n}{\sqrt{n!}} \mid n\rangle \; \equiv | z e^{-i\omega(t-t_0)}\rangle \; . \tag{72}$$

i.e.: if a $t_0 \mid \Psi\rangle$ is coherent state for z then at t it becomes coherent state for $z \exp[-i\omega(t - t_0)]$ (periodic with ω)

Use the notation

$$z(t) = \; z \exp[-i\omega(t - t_0)] \tag{73}$$

Overcompleteness of coherent states

use the notation: $z \;=\; z_r \;+\; i\; z_i$. Then define

$$\mathcal{I} \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mid z\rangle\langle z \mid d\; z_r \; d\; z_i \tag{74}$$

Coherent states V

and the matrix elements

$$\mathcal{I}_{nm} = \frac{1}{\sqrt{n!\ m!}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (z^*)^n\ z^m\ e^{-|z|^2} d\ z_r\ d\ z_i\ .$$

Calculate in polar coordinates $z_r = \rho\ \cos\phi,\ \ z_i = \rho\ \sin\phi$, and one obtains $\mathcal{I}_{nm} = \pi \delta_{nm}$, i.e.

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mid z \rangle \langle z \mid d\ z_r\ d\ z_i\ = \ \hat{I}\ , \tag{75}$$

A coherent state can be written as a superposition of coherent states. Proof: one starts from $\mid z' \rangle = \hat{I}\ \mid z' \rangle$,

$$\mid z' \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle z \mid z' \rangle\ \mid z \rangle\ d\ z_r\ d\ z_i\ . \tag{76}$$

The coherent states are an overcomplete set. Also

- The expansion of an arbitrary vector in coherent states is not unique

Coherent states VI

- the subset

$$\{| \, n_1 \; + \; i \; n_2 \, \rangle\}_{n_1, n_2 \in Z} \; ,$$

is complete

The electromagnetic field in a coherent state: the expectation value of the electric field

$$\begin{aligned} \overline{\mathbf{E}(t)} \quad &\equiv \quad < z(t) \mid \mathbf{E}^{\mathrm{op}}(\mathbf{r}) \mid z(t) > \\ &= \quad \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \; i \, [\, z \, e^{i(\boldsymbol{\kappa}\cdot\mathbf{r} - \omega(t - t_0))} \, \mathbf{s} \; - \; z^* \, e^{-i(\boldsymbol{\kappa}\cdot\mathbf{r} - \omega(t - t_0))} \, \mathbf{s}^* \,]. \end{aligned} \tag{77}$$

(like in the classical description). Similar for the magnetic field. By direct calculation the standard deviation of E

$$\Delta E \; = \; \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \; , \tag{78}$$

constant, the same for the vacuum state.

Coherent states are "close" the classical states

Coherent states VII

Generation of coherent states

$$| z\rangle = e^{-\frac{1}{2}|z|^2} \; e^{z\hat{a}^\dagger} \; | 0\rangle \, . \tag{79}$$

One defines

$$\hat{D}(z) \equiv e^{z\hat{a}^\dagger - z^* \hat{a}} \; . \tag{80}$$

(Glauber displacement operator) with

$$\hat{D}(z) = e^{-\frac{1}{2}|z|^2} \; e^{z\hat{a}^\dagger} \; e^{-z^* \hat{a}} \; , \tag{81}$$

$$\hat{D}(z) \mid 0\rangle = e^{z\hat{a}^\dagger - z^* \hat{a}} \mid 0\rangle = \mid z\rangle \, , \tag{82}$$

$$\hat{D}(z)\, \hat{a}\, \hat{D}(z)^\dagger \;=\; \hat{a} - z\hat{I} \, , \quad \hat{D}(0) = \hat{I} \, , \qquad \hat{D}(z)^\dagger = \hat{D}(-z) \, . \tag{83}$$

Minimum uncertainity states: one defines

$$\hat{Q} \equiv \left(\frac{\hbar}{2\omega}\right)^{1/2} \left(\hat{a}^\dagger \; + \; \hat{a} \right) \, , \quad \hat{P} \equiv i \left(\frac{\hbar\omega}{2}\right)^{1/2} \left(\hat{a}^\dagger \; - \; \hat{a} \right) \, . \tag{84}$$

Coherent states VIII

with

$$[\,\hat{Q}\,,\,\hat{P}\,] \;=\; i\,\hbar\hat{I}\,, \tag{85}$$

i.e. "position" and "momentum" operators

$$\begin{aligned} \langle z \mid \hat{Q} \mid z\rangle &= \frac{2\hbar}{\omega}\,\mathrm{Re}\,(z)\,, \quad \langle z \mid \hat{P} \mid z\rangle = 2\hbar\omega\,\mathrm{Im}\,(z)\,, \\ \overline{(\delta Q)^2} &= \hbar/2\omega\,, \qquad\qquad \overline{(\delta P)^2} = \hbar\omega/2\,, \end{aligned} \tag{86}$$

and $\Delta Q \cdot \Delta P \;=\; \hbar/2$, (minimum)

Phase operator I

One defines

$$\hat{e} = \sum_{m=0}^{\infty} |\, m\rangle\langle m+1 \,| \qquad \hat{e}^{\dagger} = \sum_{m=0}^{\infty} |\, m+1\rangle\langle m \,| \ . \tag{87}$$

Properties:

$$\hat{e} \mid n\rangle \ = \ \sum_{m=0}^{\infty} |\, m\rangle \delta_{m,n-1} = |\, n-1\rangle \ , \quad n \neq 0 \ , \tag{88}$$

$$\hat{e} \mid 0\rangle \ = \ |\, \mathrm{zero}\rangle \ , \quad \hat{e}^{\dagger} \mid n\rangle = |\, n+1\rangle \ , \tag{89}$$

$$\hat{e}\hat{e}^{\dagger} = \hat{I} \ , \quad \hat{e}^{\dagger}\hat{e} = \hat{I} - \ |\, 0\rangle\langle 0 \,| \ . \tag{90}$$

(isometric, *not* unitary). Also

$$[\hat{e}, \hat{e}^{\dagger}] = |\, 0\rangle\langle 0 \,| \ , \qquad [\hat{e}, \hat{N}] = \hat{e} \ . \tag{91}$$

One can write

$$\hat{a} \ = \ (\hat{N} + \hat{I})^{1/2} \, \hat{e} \ , \quad \hat{a}^{\dagger} \ = \ \hat{e}^{\dagger} (\hat{N} + \hat{I})^{1/2} \ . \tag{92}$$

Phase operator II

or

$$\hat{C} \equiv \frac{1}{2}[\,\hat{e} \;+\; \hat{e}^{\dagger}\,]\,, \quad \hat{S} \equiv \frac{1}{2i}\,[\,\hat{e} \;-\; \hat{e}^{\dagger}\,]\,. \tag{93}$$

One can write

$$\hat{e} = e^{i\hat{\phi}} \tag{94}$$

and

$$\hat{C} = \cos\hat{\phi}, \quad \hat{S} = \sin\hat{\phi} \tag{95}$$

Expectation values in a Fock state is zero; in a coherent state non-zero.

Uncertainity relations in a arbitrary state:

$$[\hat{C}, \hat{N}] = i\;\hat{S}\;, \quad [\hat{S}, \hat{N}] = -\;i\;\hat{C}\;, \quad [\hat{S}, \hat{C}] = \frac{i}{2}\;\;|\,0\rangle\langle 0\,| \tag{96}$$

$$\Delta C \cdot \Delta N \geq \frac{1}{2}\,|\,\overline{S}\,|\;, \quad \Delta S \cdot \Delta N \geq \frac{1}{2}\,|\,\overline{C}\,|\;, \quad \Delta C \cdot \Delta S \geq \frac{1}{4}\,|\,\overline{|\,0\rangle\langle 0\,|}\;\,|\;\;. \tag{97}$$

Phase operator III

The eigenvalue problem of $\hat{e}$:

$$\hat{e} \mid \phi\rangle = e^{i\phi} \mid \phi\rangle \tag{98}$$

$$\mid \phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\phi} \mid n\rangle \ . \tag{99}$$

$$\langle\phi \mid \phi'\rangle \ = \ \delta(\phi - \phi') \quad \int_0^{2\pi} \mid \phi\rangle\langle\phi \mid d\phi \ = \ \hat{I} \ . \tag{100}$$

Semiclassical approximation

Work in Heisenberg picture: In a coherent state the expectation values of E and B concide with the classical expressions. Use the unitary transformation

$$\hat{D}(z)^{\dagger}\mathbf{A}^{op,H}(\mathbf{r},t)\hat{D}(z) = \sqrt{\frac{\hbar}{2\epsilon_0 V\omega}}\left[\mathbf{s}\, z\, e^{i(\boldsymbol{\kappa}\cdot\mathbf{r}-\omega t)} + \mathbf{s}^{*} z^{*} e^{-i(\boldsymbol{\kappa}\cdot\mathbf{r}-\omega t)}\right] + \mathbf{A}^{op,H}(\mathbf{r},t)\,. \tag{101}$$

$$\mid\Psi\rangle \rightarrow \hat{D}(z)^{\dagger}\mid\Psi\rangle\,. \tag{102}$$

If $\mid\Psi\rangle = \mid z\rangle$ then $\hat{D}(z)^{\dagger}\mid\Psi\rangle = |0\rangle$ and the expectation values of the second term in the transformed operator is zero.

More general: semicalssical description is “good” for large number of photons.

NR dipole approximation I

Use velocity gauge: $\mathbf{A} \equiv \mathbf{A}(t)$ and $\Phi = 0$

$$\mathbf{A}(t) = f(t)\left[\zeta_x \mathbf{e_x} \sin(\omega t) + \zeta_y \mathbf{e_y} \cos(\omega t)\right], \quad \zeta_x^2 + \zeta_y^2 = 1 \tag{103}$$

with $f(t)$ arbitrary, and $\lim_{t \pm \infty} f(t) = 0$

$$\mathbf{E}(t) = -\frac{dA(t)}{dt}, \quad \mathbf{B}(t) = 0 \tag{104}$$

$$m\ddot{\mathbf{r}}(t) = eE(t), \quad m\mathbf{r}(t_0) = 0, \quad m\dot{\mathbf{r}}(t_0) = \mathbf{v}_0 \tag{105}$$

Solution

$$m\dot{\mathbf{r}}(t) = -e\mathbf{A}(t) + \mathbf{v}_0 \tag{106}$$

$$\mathbf{r}(t) = -\frac{e}{m}\int_{t_0}^{t} \mathbf{A}(t')dt' + \mathbf{v}_0(t - t_0) \tag{107}$$

Notation

$$-\frac{e}{m}\int_{t_0}^{t} \mathbf{A}(t')dt' = \boldsymbol{\alpha}(t) \tag{108}$$

NR non-dipole approximation I

Consider *only* plane-wave fields: $A = A(\phi) = A(ct - z)$. Initial conditions:

We shall assume that at a given moment of time t_0 the particle is in the origin of the reference frame, and it has the given velocity $\mathbf{v}_0$; in the relativistic case the corresponding four-velocity will be denoted by u_0, and the four-momentum by p_0. The field propagates along the **n** direction, chosen parallel to Oz, and is described by the potential $A(\phi) \equiv A(ct - z)$. We denote by A_0 the value of the potential A in the origin of the reference frame at the initial moment t_0

$$A_0 \equiv (0, \mathbf{A_0}) = A(\phi_0) \tag{109}$$

with $\phi_0 = ct_0$. For the case of a pulse, the most natural choice for t_0 is a moment sufficiently far in the past, when the pulse has not reached the origin and the particle is free (i.e. $A_0 = 0$).

Cases of interest:

Case 1: linearly polarized pulse; the particle is at rest in the origin at a moment t_0 very far in the past, when the pulse has not yet reached the origin.

NR non-dipole approximation II

Case 2: linearly polarized monochromatic field; at the moment t_0 when the field vanishes in the origin, the particle is located in the origin, with an initial velocity $\mathbf{v}_0$ directed along the field propagation direction.

The Lagrange function of a nonrelativistic particle of mass m and electric charge e moving in the electromagnetic field $A(\phi)$

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{m\dot{\mathbf{r}}^2}{2} + e\dot{\mathbf{r}} \cdot \mathbf{A}(\phi) \tag{110}$$

gives the equations of motion

$$\frac{d}{dt}\left(m\dot{\mathbf{r}}_\perp + e\mathbf{A}(\phi)\right) = 0, \tag{111}$$

$$\frac{d}{dt}(m\dot{z}) = e\dot{\mathbf{r}} \cdot \frac{\partial \mathbf{A}(\phi)}{\partial z} = -e\dot{\mathbf{r}} \cdot \frac{d\mathbf{A}(\phi)}{d\phi}, \tag{112}$$

NR non-dipole approximation III

where $\mathbf{r}_\perp$ is the component of $\mathbf{r}$ orthogonal on the propagation direction $\mathbf{n} \equiv \mathbf{e}_z$. From the first of the two equations above, and taking into account the initial conditions one obtains

$$\dot{\mathbf{r}}_\perp = \mathbf{v}_{0\perp} - \frac{e}{m}(\mathbf{A}(\phi) - \mathbf{A}_0). \tag{113}$$

Using this result in Eq.(112) one gets

$$\frac{d}{dt}\dot{z} = \frac{d}{d\phi}\left[\frac{e^2}{2m^2}\left(\mathbf{A}(\phi) - \mathbf{A}_0 - \frac{m}{e}\mathbf{v}_{0\perp}\right)^2\right] \tag{114}$$

It is convenient to use ϕ as the independent variable. From the relation

$$\frac{d\phi}{dt} = c\left(1 - \frac{\dot{z}}{c}\right) \tag{115}$$

one can see that this change of variable only makes sense if $\dot{z} < c$; the restriction is however not a problem, since we are not interested, anyway, to find a solution with $\dot{z} \geq c$. The above equation in the new variable ϕ writes as

$$\frac{d}{d\phi}\left(\frac{\dot{z}}{c} - \frac{1}{2}\frac{\dot{z}^2}{c^2}\right) = \frac{d}{d\phi}\left[\frac{e^2}{2(mc)^2}\left(\mathbf{A}(\phi) - \mathbf{A}_0 - \frac{m}{e}\mathbf{v}_{0\perp}\right)^2\right]. \tag{116}$$

NR non-dipole approximation IV

Taking into account the initial conditions one obtains the equation

$$\frac{\dot{z}}{c} - \frac{1}{2}\frac{\dot{z}^2}{c^2} = \frac{e^2\boldsymbol{\mathcal{A}}^2(\phi)}{2(mc)^2} + \frac{v_{03}}{c} - \frac{1}{2}\frac{v_{03}^2}{c^2}, \tag{117}$$

with

$$\boldsymbol{\mathcal{A}}^2(\phi) = (\mathbf{A}(\phi) - \mathbf{A}_0)^2 - \frac{2m\mathbf{v}_{0\perp}}{e} \cdot (\mathbf{A}(\phi) - \mathbf{A}_0) \tag{118}$$

whose solution is

$$\frac{\dot{z}}{c} = 1 - \sqrt{\left(1 - \frac{v_{03}}{c}\right)^2 - \frac{e^2\boldsymbol{\mathcal{A}}^2(\phi)}{(mc)^2}}. \tag{119}$$

One notices that the solution $\dot{z}$ is defined only if

$$\frac{e^2\boldsymbol{\mathcal{A}}^2(\phi)}{2(mc)^2} < \left(1 - \frac{v_{03}}{c}\right)^2 \tag{120}$$

NR non-dipole approximation V

and that $\dot{z}$ cannot become larger than c. Again, using ϕ as the independent variable, one gets

$$\frac{dz}{d\phi} = \frac{1}{c - \frac{dz}{dt}} \frac{dz}{dt}, \qquad \frac{d\mathbf{r}_\perp}{d\phi} = \frac{1}{c - \frac{dz}{dt}} \frac{d\mathbf{r}_\perp}{dt} \tag{121}$$

or

$$z = \int_{\phi_0}^{ct-z} d\chi \frac{1 - \sqrt{\left(1 - \frac{v_{03}}{c}\right)^2 - \frac{e^2 \boldsymbol{\mathcal{A}}^2(\chi)}{(mc)^2}}}{\sqrt{\left(1 - \frac{v_{03}}{c}\right)^2 - \frac{e^2 \boldsymbol{\mathcal{A}}^2(\chi)}{(mc)^2}}}, \qquad \phi_0 = ct_0 \tag{122}$$

$$\mathbf{r}_\perp = -\frac{e}{mc} \int_{\phi_0}^{ct-z} d\chi \frac{\mathbf{A}(\chi) - \mathbf{A}_0 - \frac{m\mathbf{v}_\perp}{e}}{\sqrt{\left(1 - \frac{v_{03}}{c}\right)^2 - \frac{e^2 \boldsymbol{\mathcal{A}}^2(\chi)}{(mc)^2}}}. \tag{123}$$

The first of the above equations must be solved for z, then its solution used in the second one, to get $\mathbf{r}_\perp$.

NR non-dipole approximation VI

Examples:

Case 1 is described by the conditions: $\phi_0 = -\infty$, $\mathbf{A}_0 = \mathbf{0}$, $\mathbf{v}_0 = \mathbf{0}$, the corresponding trajectory being:

$$z = \int_{-\infty}^{ct-z} d\chi \frac{1 - \sqrt{1 - \frac{e^2\mathbf{A}^2(\chi)}{(mc)^2}}}{\sqrt{1 - \frac{e^2\mathbf{A}^2(\chi)}{(mc)^2}}}, \qquad \mathbf{r}_\perp = -\frac{e}{mc}\int_{\phi_0}^{ct-z} d\chi \frac{\mathbf{A}(\chi)}{\sqrt{1 - \frac{e^2\mathbf{A}^2(\chi)}{(mc)^2}}}. \tag{124}$$

The solution is defined for $e^2\mathbf{A}^2(\phi) < (mc)^2$; in fact, the non-relativistic approximation is valid only in the limit $e^2\mathbf{A}^2(\phi) \ll (mc)^2$. In this case the previous solution becomes

$$z = \int_{-\infty}^{ct-z} d\chi \frac{e^2\mathbf{A}^2(\chi)}{2(mc)^2}, \qquad \mathbf{r}_\perp = -\frac{e}{mc}\int_{\phi_0}^{ct-z} d\chi \mathbf{A}(\chi). \tag{125}$$

In the **case 2** we have $\phi_0 = 0$, $\mathbf{A}_0 = \mathbf{0}$, $\mathbf{v}_0 = v_{03}\,\mathbf{n}$, and the corresponding solution

$$z = \int_{0}^{ct-z} d\chi \frac{1 - \sqrt{\left(1 - \frac{v_{03}}{c}\right)^2 - \frac{e^2\mathbf{A}^2(\chi)}{(mc)^2}}}{\sqrt{\left(1 - \frac{v_{03}}{c}\right)^2 - \frac{e^2\mathbf{A}^2(\chi)}{(mc)^2}}} \tag{126}$$

NR non-dipole approximation VII

$$\mathbf{r}_\perp = -\frac{e}{mc}\int\limits_0^{ct-z} d\chi \frac{\mathbf{A}(\chi)}{\sqrt{\left(1-\frac{v_{03}}{c}\right)^2 - \frac{e^2\mathbf{A}^2(\chi)}{(mc)^2}}}. \tag{127}$$

In this case the conditions of validity of the non-relativistic approximation are $e^2\mathbf{A}^2(\phi) \ll (mc)^2$ and $v_{03} \ll c$, which leads to the equations of motion

$$z = \int\limits_{-\infty}^{ct-z} d\chi \left[\frac{v_{03}}{c} + \frac{e^2\mathbf{A}^2(\chi)}{2(mc)^2}\right], \qquad \mathbf{r}_\perp = -\frac{e}{mc}\int\limits_{\phi_0}^{ct-z} d\chi \mathbf{A}(\chi). \tag{128}$$

Relativistic case I

Plane wave; the same initial conditions and particular cases as previously.

In the relativistic case the equations of motions are

$$\frac{dp^{\mu}}{d\tau} = eF^{\mu\nu}u_{\nu}, \qquad \underline{u} = \frac{d\underline{x}}{d\tau}, \qquad \underline{p} = m\underline{u}, \tag{129}$$

where τ is the proper time and $\hat{\mathbf{F}}$ is the electromagnetic four-tensor of the field intensities:

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} \equiv \begin{Vmatrix} 0 & \frac{dA^1(\phi)}{d\phi} & \frac{dA^2(\phi)}{d\phi} & 0 \\ -\frac{dA^1(\phi)}{d\phi} & 0 & 0 & -\frac{dA^1(\phi)}{d\phi} \\ -\frac{dA^2(\phi)}{d\phi} & 0 & 0 & -\frac{dA^2(\phi)}{d\phi} \\ 0 & \frac{dA^1(\phi)}{d\phi} & \frac{dA^2(\phi)}{d\phi} & 0 \end{Vmatrix}. \tag{130}$$

Relativistic case II

With the above definitions the equations of motion become

$$\frac{dp^0}{d\tau} = \frac{dp^3}{d\tau} = -e\frac{d\mathbf{A}(\phi)}{d\phi}\cdot\mathbf{u}_\perp\,, \tag{131}$$

$$\frac{d\mathbf{p}_\perp}{d\tau} = -e\frac{d\mathbf{A}(\phi)}{d\phi}\left(u^0 - u^3\right). \tag{132}$$

From the first of the above formulae one obtains

$$\frac{d(p^0 - p^3)}{d\tau} = 0 \tag{133}$$

i.e. $p^0 - p^3 = m(u^0 - u^3)$ is a constant of motion; taking into account the initial conditions we have $p^0 - p^3 = m(u_0^0 - u_0^3)$. Further, noticing that

$$u^0 - u^3 = \mathrm{const} = \frac{d}{d\tau}\left(x^0 - x^3\right) = \frac{d\phi}{d\tau} \tag{134}$$

Eq.(132) becomes

$$\frac{d\mathbf{p}_\perp}{d\tau} = -e\,\frac{d\mathbf{A}}{d\phi}\,\frac{d\phi}{d\tau}. \tag{135}$$

Relativistic case III

It is again convenient to look for the solutions of the above equations as functions of ϕ. Using the initial conditions one obtains

$$\mathbf{p}_\perp(\phi) = -e\left(\mathbf{A}(\phi) - \mathbf{A}_0 + \frac{\mathbf{p}_{0\perp}}{e}\right), \tag{136}$$

and, then, from Eqs.(131) and (136)

$$\frac{dp^3(\phi)}{d\phi} = \frac{e^2}{2m(u_0^0 - u_0^3)}\frac{d}{d\phi}\left(\mathbf{A}(\phi) - \mathbf{A}_0 + \frac{\mathbf{p}_{0\perp}}{e}\right)^2 \tag{137}$$

or

$$p^3(\phi) = \frac{e^2\boldsymbol{\mathcal{A}}^2(\phi)}{2m(u_0^0 - u_0^3)} + p_{03}, \tag{138}$$

with

$$\boldsymbol{\mathcal{A}}^2(\phi) = (\mathbf{A}(\phi) - \mathbf{A}_0)^2 - \frac{2\mathbf{p}_{0\perp}}{e}\cdot(\mathbf{A}(\phi) - \mathbf{A}_0). \tag{139}$$

Relativistic case IV

The differential equations in ϕ obeyed by the coordinates are

$$\frac{d\mathbf{r}}{d\phi} = \frac{1}{(u_0^0 - u_0^3)} \frac{d\mathbf{r}}{d\tau} = \frac{\mathbf{p}(\phi)}{m(u_0^0 - u_0^3)} \tag{140}$$

with the solution

$$\mathbf{r}_\perp = -\frac{e}{m(u_0^0 - u_0^3)} \int_{\phi_0}^{ct-z} \left[\mathbf{A}(\chi) - \mathbf{A}_0 - \frac{\mathbf{p}_{0\perp}}{e}\right] d\chi \tag{141}$$

$$z = \int_{\phi_0}^{ct-z} d\chi \left[\frac{e^2 \boldsymbol{\mathcal{A}}^2(\chi)}{2m^2(u_0^0 - u_0^3)^2} + \frac{u_0^3}{u_0^0 - u_0^3}\right]. \tag{142}$$

Relativistic case V

Examples: In the particular **case 1** the relativistic solution reduces to

$$\mathbf{r}_\perp = -\frac{e}{mc}\int_{-\infty}^{ct-z}\mathbf{A}(\chi)d\chi, \quad z = \frac{e}{2(mc)^2}\int_{-\infty}^{ct-z}\mathbf{A}^2(\chi)d\chi. \tag{143}$$

In the low intensity limit the above equation become identical with the non-relativistic result. Usually the total displacement in the polarization plane vanishes, so that at the end of the pulse the particle is left at rest in a point along the Oz axis. If the envelope is simple enough it is also possible to calculate the total displacement along the propagation direction; for example, for a Gaussian pulse of amplitude A_0, frequency ω and FWHM τ_p one gets

$$\Delta z = \frac{e^2 A_0^2}{4m^2 c}\sqrt{\frac{\pi}{2}}\frac{T\tau_p}{1.1774}\left[1 - \exp\left[-\frac{1}{2}\left(\frac{2\pi\tau_p}{1.1774}\right)^2\right]\right] \tag{144}$$

Relativistic case VI

For all realistic cases the second term in the previous equation is negligible, and the total displacement becomes

$$\Delta z = \frac{e^2 A_0^2}{4m^2 c}\sqrt{\frac{\pi}{2}}\frac{T\tau_p}{1.1774}. \tag{145}$$

In the **case 2** the solution is

$$\mathbf{r}_\perp = -\frac{e}{m(u_0^0 - u_0^3)}\int_0^{ct-z} d\chi \mathbf{A}(\chi) \tag{146}$$

$$z = \int_0^{ct-z} d\chi \left[\frac{e^2\mathbf{A}^2(\chi)}{2m^2(u_0^0 - u_0^3)^2} + \frac{u_0^3}{u_0^0 - u_0^3}\right]. \tag{147}$$

If the initial velocity $\mathbf{u}_0$ vanishes the trajectory has a "Z" like shape - an oscillation along the polarization direction, composed with an advance along the propagation direction. It

Relativistic case VII

is possible to find a particular "initial" velocity $\mathbf{v}_0$ such that the total displacement along the Oz axis during one optical period cancels. The condition writes as

$$\frac{e^2 A_0^2(\phi)}{4m^2(u_0^0 - u_0^3)^2} + \frac{u_0^3}{(u_0^0 - u_0^3)} = 0 \tag{148}$$

and it is satisfied for the initial velocity

$$\mathbf{v}_0 = -\mathbf{n}c\,\frac{e^2 A_0^2}{4(mc)^2 + e^2 A_0^2} \equiv \mathbf{n}V_0 \tag{149}$$

With this initial condition, and using the notation

$$m^{*2} = m^2 + \frac{e^2 A_0^2}{2c^2} \tag{150}$$

the solution becomes

$$\mathbf{r}_\perp = -\frac{e}{m^* c}\int_0^{ct-z} d\chi \mathbf{A}(\chi), \qquad z = \frac{e^2}{2(m^* c)^2}\int_0^{ct-z} d\chi \left[\mathbf{A}^2(\chi) - \frac{A_0^2}{2}\right]. \tag{151}$$

Relativistic case VIII

In this case the trajectory has the well known "figure 8" shape in the plane Oxz. Is important to notice that, unlike in the **case 1** the amplitude of the oscillation is limited along both Ox and Oz directions when A_0 tends to infinity; the limits are

$$\lim_{A_0 \to \infty} \Delta x = \sqrt{2}\frac{c}{\omega}, \qquad \lim_{A_0 \to \infty} \Delta z = \frac{c}{4\omega}. \tag{152}$$

The quantity m^* defined in Eq.(150) is the so-called "dressed mass"; it is an average effective mass of the electron interacting with the monochromatic field.

Numerical examples I

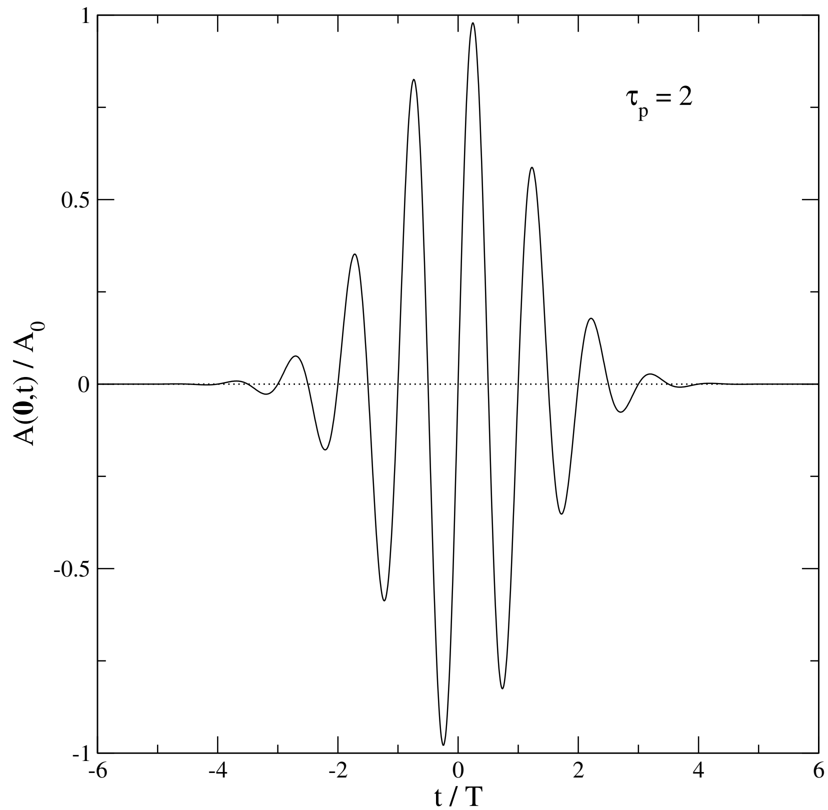


Figure: The vector potential **A** in the origin of the reference frame, for the case of a linearly polarized Gaussian pulse of $\tau_p = 2$, $\omega = 1$ au.

Numerical examples II

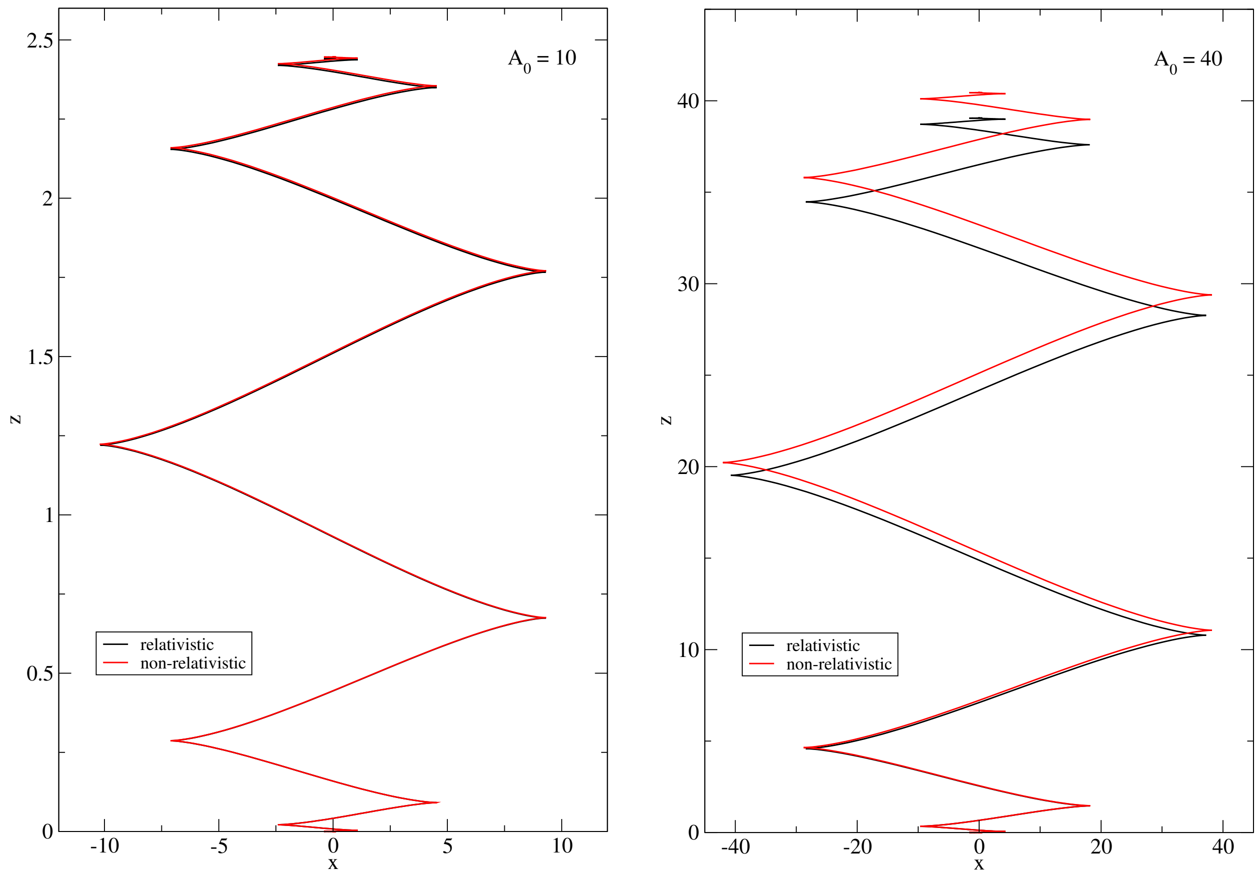


Figure: **Case 1**: the relativistic (in black) and non-relativistic (in red) trajectories for a lineraly polarized pulse with $\tau_p = 2$ cycles, $\omega = 1$ au and for three values of the amplitude A_0.

Numerical examples III

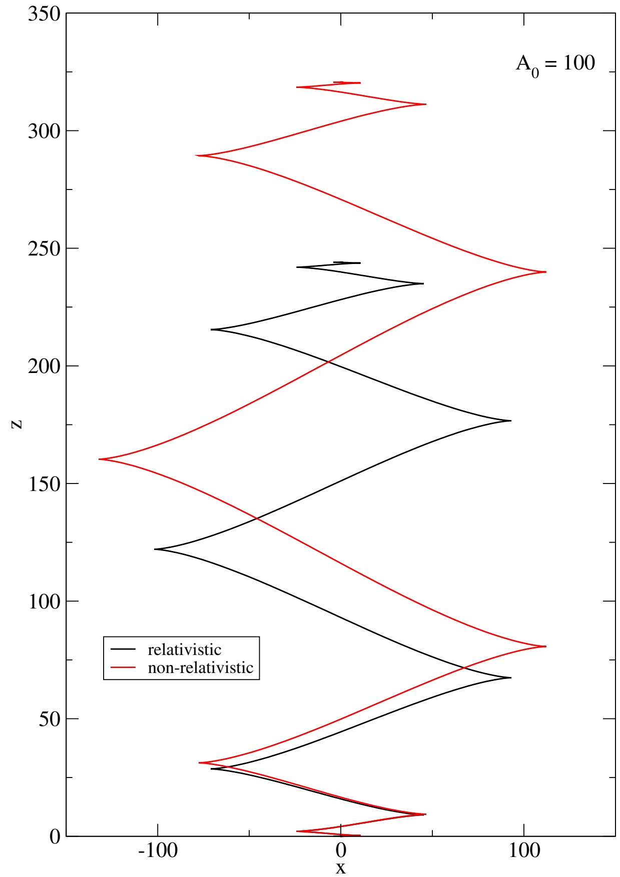


Figure: **Case 1**: the relativistic (in black) and non-relativistic (in red) trajectories for a lineraly polarized pulse with $\tau_p = 2$ cycles, $\omega = 1$ au and for three values of the amplitude A_0.

Numerical examples IV

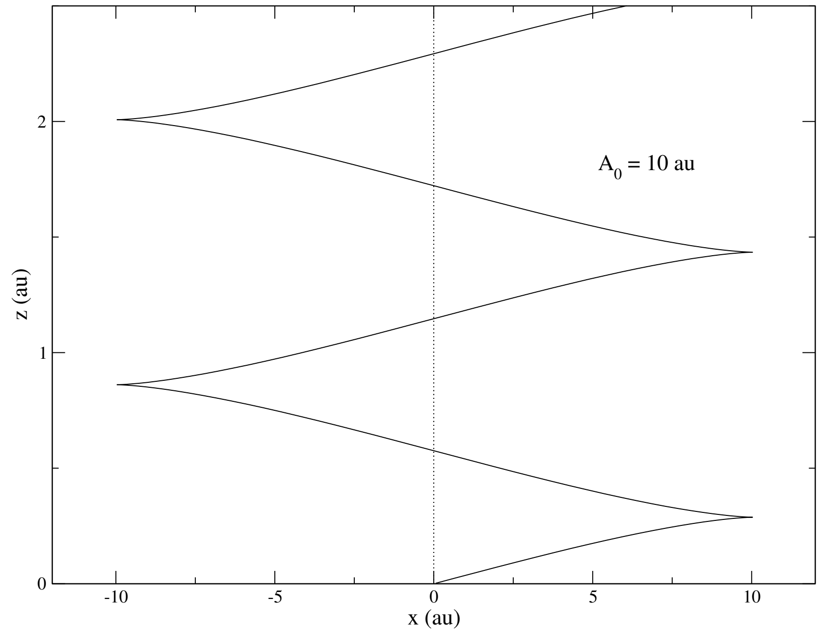


Figure: **Case 2**, initial condition $\mathbf{v}_0 = 0$: the relativistic trajectory for monochromatic linearly polarized field of amplitude $A_0 = 10$ au.

Numerical examples V

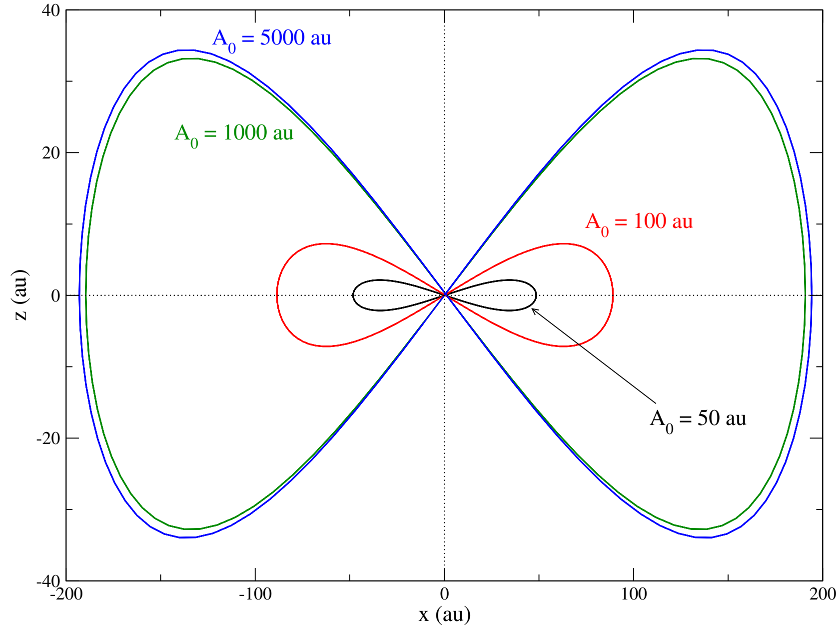


Figure: **Case 2**, initial condition $\mathbf{v}_0 = V_0\mathbf{n}$: the relativistic trajectories for monochromatic linearly polarized field and four values of the amplitude A_0.

Important conclusions: for typical cases, *plane wave*

- the laser field: finite pulse

Numerical examples VI

- dipole approximation: the position and velocity of the electron at the end of the pulse is the same as at the initia moment
- relativistic corrections: the electron velocity at the end of the pulse is the same as at the initial moment, i.e. no net energy gain
- relativistic corrections: there is a net displacement along the pulse propagation direction

Derivation of Dirac Volkov solutions I

D. M. Volkov (1953)

Plane wave field:

$$A(x) \equiv (A_0(ct - \mathbf{n}\cdot\mathbf{r}), \mathbf{A}(ct - \mathbf{n}\cdot\mathbf{r})) \equiv A(\phi). \qquad (153)$$

with $\phi = ct - \mathbf{n}\cdot x$. Define $n = (1, \mathbf{n})$, $n\cdot n = 1 - \mathbf{n}^2 = 0$, $x \equiv (ct, \mathbf{r})$

$$\phi = ct - \mathbf{n}\cdot\mathbf{r} = n\cdot x \qquad (154)$$

Gauge

$$\partial_\mu\phi\frac{dA^\mu}{d\phi} = n_\mu\frac{dA^\mu}{d\phi} = \frac{d(n\cdot A)}{d\phi} = 0 \qquad (155)$$

i.e. $n\cdot A = const$. Choice: $n\cdot A = 0$.

Choice of the reference frame:

- $\mathbf{n} = \mathbf{e}_z$,
- $\mathbf{a} = \mathbf{a}_\perp + \mathbf{a}_\parallel$, $\quad \mathbf{a}_\parallel = \mathbf{n}(\mathbf{n}\cdot\mathbf{a})$, $\quad \mathbf{a}_\perp = \mathbf{n}\times(\mathbf{a}\times\mathbf{n})$

Derivation of Dirac Volkov solutions II

Dirac equation:

$$[\gamma^\mu (P_\mu - eA_\mu) - mc]\,\psi(x) = 0, \tag{156}$$

$$\gamma^0 = \beta, \quad \gamma^i = \beta\alpha_i \tag{157}$$

and

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{158}$$

The usual notation

$$\Pi_\mu = P_\mu - eA_\mu \tag{159}$$

$$\left[\widehat{\Pi} - mc\right]\psi(x) = 0, \tag{160}$$

Look for the solution

$$\psi_i(p;x) = \left[\widehat{\Pi} + mc\right]\exp\left[-\frac{\mathrm{i}}{\hbar}\,\epsilon_i\,(p\cdot x)\right]\Phi_i(\phi;p)\mathcal{Z}_i \qquad i = 1,\ldots,4 \tag{161}$$

Derivation of Dirac Volkov solutions III

$p\cdot p=(mc)^2$, $\epsilon_i=\begin{cases}+1, & i=1,2\\ -1, & i=3,4\end{cases}$, $\mathcal{Z}_i$: constant

$$\left[\widehat{\Pi}-mc\right]\left[\widehat{\Pi}+mc\right]\exp\left[-\frac{\mathrm{i}}{\hbar}\epsilon_i\left(p\cdot x\right)\right]\Phi_i(\phi;p)\mathcal{Z}_i=0 \qquad (162)$$

$$\begin{aligned}\widehat{A}\widehat{B}+\widehat{B}\widehat{A} &= A_\mu B_\nu\gamma^\mu\gamma^\nu+B_\nu A_\mu\gamma^\nu\gamma^\mu\\ &= A_\mu B_\nu(\gamma^\nu\gamma^\mu+\gamma^\mu\gamma^\nu)+[B_\nu,A_\mu]\gamma^\nu\gamma^\mu\\ &= 2A\cdot B+C_{\mu\nu}\gamma^\mu\gamma^\nu \qquad (163)\end{aligned}$$

$$[\Pi_\mu,\Pi_\nu]=\mathrm{i}e\hbar\left(\partial_\mu A_\nu(\phi)-\partial_\nu A_\mu(\phi)\right)=\mathrm{i}e\hbar\left(n_\mu\frac{dA_\nu}{d\phi}-n_\nu\frac{dA_\mu}{d\phi}\right), \qquad (164)$$

One obtains

$$\widehat{\Pi}\widehat{\Pi}=\Pi^2+\frac{1}{2}\mathrm{i}e\hbar\widehat{n}\frac{d\widehat{A}}{d\phi}-\frac{1}{2}\mathrm{i}e\hbar\frac{d\widehat{A}}{d\phi}\widehat{n}=(P^2-2eA(\phi)\cdot P+e^2A^2(\phi))+\mathrm{i}e\hbar\widehat{n}\frac{d\widehat{A}}{d\phi}; \qquad (165)$$

Derivation of Dirac Volkov solutions IV

(NB: $\{\widehat{n}, \widehat{A}\} = 0$, $A \cdot P = P \cdot A$)

$$-2\mathrm{i}\hbar\epsilon_i(n \cdot p)\frac{d\Phi_i(\phi)}{d\phi} + \left[\mathrm{i}e\hbar\widehat{n}\frac{d\widehat{A}(\phi)}{d\phi} + e^2A^2(\phi) - 2\epsilon_i eA(\phi)\cdot p\right]\Phi_i(\phi) = 0, \quad (166)$$

$$\frac{d\Phi_i}{d\phi} = \left[-\frac{\mathrm{i}}{\hbar}\frac{1}{2n\cdot p}(\epsilon_i A^2(\phi) - 2eA(\phi)\cdot p) + \frac{\epsilon_i e}{2n\cdot p}\widehat{n}\frac{d\widehat{A}}{d\phi}\right]\Phi_i. \quad (167)$$

(NB: differential equation for a *matrix*)

$$\Phi_i = \exp\left[-\frac{\mathrm{i}}{\hbar}\frac{1}{2(n\cdot p)}\int^{\phi}\left[\epsilon_i e^2 A^2(\chi) - 2eA(\chi)\cdot p\right]d\chi\right]\exp\left[\frac{\epsilon_i e}{2n\cdot p}\widehat{n}\widehat{A}(\phi)\right]. \quad (168)$$

$$\left[\widehat{n}\cdot\widehat{A}(\phi)\right]^2 = -\widehat{A}(\phi)\widehat{n}\widehat{n}\widehat{A}(\phi) = 0 \quad (169)$$

$$\Phi_i(p;x) = \left[1 + \epsilon_i\frac{e}{2(n\cdot p)}\widehat{n}\widehat{A}(\phi)\right]\exp[\Lambda_i(\phi;p)] \qquad i = 1,\ldots,4 \quad (170)$$

Derivation of Dirac Volkov solutions V

$$\Lambda_i(\phi; p) = -\frac{\mathrm{i}}{\hbar}\frac{1}{2(n\cdot p)}\int^{\phi}\left[2eA(\chi)\cdot p - \epsilon_i\, e^2 A^2(\chi)\right] d\chi. \tag{171}$$

The solution $\psi_i(p, x)$

$$\begin{aligned}\psi_i(p; x) &= \left[\widehat{\Pi} + mc\right] \times \\ &\times \exp\left[-\frac{\mathrm{i}}{\hbar}\epsilon_i\,(p\cdot x) + \Lambda_i(\phi, p)\right]\left[1 + \epsilon_i\,\frac{e}{2(n\cdot p)}\widehat{n}\widehat{A}(\phi)\right]\mathcal{Z}_i\end{aligned} \tag{172}$$

$$\psi_i(p; x) = \exp\left[-\frac{\mathrm{i}}{\hbar}\epsilon_i\,(\underline{x}\cdot\underline{p}) + \Lambda_i(\phi; \underline{p})\right]\Omega_i(\phi; \underline{p})\,(\epsilon_i\widehat{p} + mc)\mathcal{Z}_i, \qquad i = 1, 2 \tag{173}$$

$$\Omega_i(\phi; p) = \left[1 - \epsilon_i\,\frac{e}{2(n\cdot p)}\widehat{A}(\phi)\widehat{n}\right], \quad (\epsilon_i\widehat{p} + mc)\mathcal{Z}_i = \xi_i(p) \tag{174}$$

Derivation of Dirac Volkov solutions VI

Notation:

$$F_i(p;x) = -\frac{\mathrm{i}}{\hbar}\epsilon_i\,(x\cdot p) + \Lambda_i(\phi;p), \tag{175}$$

Identity:

$$\Omega_i(\phi;\underline{p})\xi_i(\underline{p}) = \left[\frac{\widehat{p} - \epsilon_i\, e\widehat{A}(\phi) + \epsilon_i\, mc}{2(\underline{n}\cdot\underline{p})}\right]\widehat{n}\xi_i(\underline{p}); \tag{176}$$

- At $t \to -\infty$ the Volkov states reduce to plane wave free states.
- positive/negative energy states: Volkov states *originating* from positive/negative energy solutinos.
- $$\mathbf{P}_\perp\psi_i(p;x) = \epsilon_i\mathbf{p}_\perp\psi_i(p;x), \tag{177}$$
  $$(n\cdot P)\psi_i(p;x) = \epsilon_i(n\cdot p)\psi_i(p;x). \tag{178}$$

Orthogonality of Dirac Volkov solutions I

Orthogonality *at the same time*

$$\langle\psi_i(p';x)|\psi_j(p;x)\rangle \equiv \int_{\mathcal{R}^3} \mathbf{dr}\,\psi_i^+(p;x)\psi_j(p';x) = \delta_{ij}\delta(\mathbf{p}-\mathbf{p}') \tag{179}$$

Proof of the orthogonality:

- (A) if both solutinos are positive (or negative) energy type
- (B) if the two are of different types

Proof for the case (A)

$$\langle\psi_i(p';x)\,|\,\psi_j(p;x)\rangle = \langle\xi_i(p')\,|\int_{\mathcal{R}^3} \mathbf{dr}\,\Omega_i^+(\phi;p')\Omega_j(\phi;p)\,\mathrm{e}^{F_i^*(p';x)+F_j(p;x)}|\,\xi_j(p)\rangle. \tag{180}$$

Orthogonality of Dirac Volkov solutions II

Direct calculation for the integral over $\mathbf{r}_\perp$

$$\langle \psi_i(p';x)\,|\,\psi_j(p;x)\rangle = (2\pi\hbar)^2\,\delta(\mathbf{p}_\perp - \mathbf{p}'_\perp)\times \\ \times\left\{\langle \xi_i(p')\,|\left[\int_{-\infty}^{\infty} dz\,\Omega_i^{+}(\phi;p')\Omega_j(\phi;p)\,\mathrm{e}^{\,Q_{ij}(z;p,p')}\right]|\,\xi_j(p)\rangle\right\}\Bigg|_{\mathbf{p}_\perp=\mathbf{p}'_\perp}, \tag{181}$$

$$Q_{ij}(z;p,p') = -\frac{\mathrm{i}}{\hbar}\left[\pm\, n\cdot(p'-p)\right]\times \\ \times\left\{\frac{1}{2}(z+ct) - \frac{1}{2(n\cdot p)(n\cdot p')}\int^{\phi}\left[(\mathbf{p}_\perp \mp e\mathbf{A}_\perp(\chi))^2 + m^2c^2\right]d\chi\right\}; \tag{182}$$

Orthogonality of Dirac Volkov solutions III

$$\Omega_i^+(\phi;p')\Omega_j(\phi;p)\Big|_{\mathbf{p}_\perp=\mathbf{p}'_\perp} = \frac{1}{4}\left(1+\frac{[\mathbf{p}_\perp \mp e\mathbf{A}_\perp(\phi)]^2+m^2c^2}{(n\cdot p)(n\cdot p')}\right)(\gamma^0\widehat{n})^2. \tag{183}$$

Change of variable

$$\rho = \frac{1}{2}\left(z+ct-\frac{1}{(n\cdot p)(n\cdot p')}\int^{\phi}\left[(\mathbf{p}_\perp \mp e\mathbf{A}_\perp(\chi))^2+m^2c^2\right]d\chi\right). \tag{184}$$

$$\langle\psi_i(p';x)|\psi_j(p;x)\rangle = (2\pi\hbar)^3\,\delta(\mathbf{p}_\perp-\mathbf{p}'_\perp)\times \\ \times\,\delta((n\cdot p')-(n\cdot p))\,\langle\xi_i(p')|(\gamma^0\widehat{n})^2|\xi_j(p)\rangle. \tag{185}$$

Use properties of free Dirac spinors

$$\langle\psi_i(p';x)|\psi_j(p;x)\rangle = \delta_{ij}\delta(\mathbf{p}-\mathbf{p}'). \tag{186}$$

Orthogonality of Dirac Volkov solutions IV

Proof for the case (B): identical calculation

$$\langle \psi_i(p';x) \,|\, \psi_j(p;x) \rangle = (2\pi\hbar)^2\, \delta(\mathbf{p}_\perp + \mathbf{p}'_\perp) \times \\ \times \left\{ \langle \xi_i(p') \,|\, \left[\int_{-\infty}^{\infty} dz\, \Omega_i^{+}(\phi;p')\Omega_j(\phi;p)\, \mathrm{e}^{\,Q_{ij}(z;p,p')} \right] |\, \xi_j(p) \rangle \right\} \Bigg|_{\mathbf{p}_\perp = -\mathbf{p}'_\perp} , \qquad (187)$$

$$Q_{ij}(z;p,p') = -\frac{\mathrm{i}}{\hbar} \left[n \cdot (p' + p) \right] \times \qquad (188) \\ \times \left\{ \frac{1}{2}(z + ct) - \frac{1}{2(n\cdot p)(n\cdot p')} \int^{\phi} \left[(\mathbf{p}_\perp - e\mathbf{A}_\perp(\chi)^2 + m^2c^2 \right] d\chi \right\};$$

Orthogonality of Dirac Volkov solutions V

$$\Omega_i^{+}(\phi;p')\Omega_j(\phi;p)\Big|_{\mathbf{p}_\perp=\mathbf{p}'_\perp} = \frac{1}{4}\left(1+\frac{[\mathbf{p}_\perp - e\mathbf{A}_\perp(\phi)]^2+m^2c^2}{(n\cdot p)(n\cdot p')}\right)(\gamma^0\widehat{n})^2. \tag{189}$$

Change of variable

$$\rho = \frac{1}{2}\left(z+ct-\frac{1}{(n\cdot p)(n\cdot p')}\int^{\phi}\left[(\mathbf{p}_\perp - e\mathbf{A}_\perp(\chi))^2+m^2c^2\right]d\chi\right). \tag{190}$$

$$\begin{aligned}\langle\psi_i(p';x)|\psi_j(p;x)\rangle &= (2\pi\hbar)^3\,\delta(\mathbf{p}_\perp+\mathbf{p}'_\perp)\times\\ &\times\delta((n\cdot p')+(n\cdot p))\,\langle\xi_i(p')|(\gamma^0\widehat{n})^2|\xi_j(p)\rangle = 0.\end{aligned} \tag{191}$$

Completeness of Dirac Volkov solutions I

$$\sum_{i=1,4} \int_{\mathcal{R}^3} d\mathbf{p}\, \psi_i(p;t,\mathbf{r})\, \psi_i^{+}(p;t,\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')\, \mathbf{I}_4. \tag{192}$$

Define

$$C(\mathbf{r},\mathbf{r}',t) \equiv \sum_{i=1,4} \int_{\mathcal{R}^3} d\mathbf{p}\, \psi_i(\mathbf{r},t;\underline{p})\, \psi_i^{+}(\mathbf{r}',t;\underline{p}) \tag{193}$$

Change of variable

$$\{\mathbf{p}_\perp, p_\parallel\} \longrightarrow \{\epsilon_i \mathbf{p}_\perp, v \equiv (\underline{n}\cdot\underline{p})\}; \tag{194}$$

$$dv = -\frac{p^0 - p^3}{p^0} dp_3, \quad p^3 = \frac{\mathbf{p}_\perp^2 + (mc)^2}{2v} - \frac{v}{2}, \quad p^0 = \frac{\mathbf{p}_\perp^2 + (mc)^2}{2v} + \frac{v}{2} \tag{195}$$

and

$$p^3 \in (-\infty,\infty), \rightarrow dv \in (\infty, 0) \tag{196}$$

Completeness of Dirac Volkov solutions II

$$C(\mathbf{r},\mathbf{r}',t) \equiv \sum_{i=1,4} \int_{\mathcal{R}^2} d\mathbf{p}_\perp \int_0^\infty \frac{dv}{v} \left(\frac{\mathbf{p}_\perp^2 + (mc)^2}{2v} + \frac{v}{2} \right) \psi_i(\mathbf{p}_\perp, v; \mathbf{r}, t)\, \psi_i^+(\mathbf{p}_\perp, v; \mathbf{r}, t). \quad (197)$$

$$C(\mathbf{r},\mathbf{r}',t) = \exp\left(\frac{\mathrm{i}}{\hbar} \int_{\phi'}^{\phi} d\chi e A_0(\chi) \right) \frac{1}{(2\pi\hbar)^3} \int_{\mathcal{R}^2} d\mathbf{p}_\perp \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathbf{p}_\perp \cdot (\mathbf{r}_\perp - \mathbf{r}'_\perp)} \times \frac{1}{2} \times \quad (198)$$

$$\times \left[\Gamma_0(\mathbf{p}_\perp, \phi, \phi') C_0(a,b) + \Gamma_{-1}(\mathbf{p}_\perp, \phi, \phi') S_{-1}(a,b) + \Gamma_{-2}(\mathbf{p}_\perp, \phi, \phi') C_{-2}(a,b) \right].$$

$$C_n(a,b) = \int_0^\infty dv\, v^n \cos\left(av - \frac{b}{v} \right), \qquad S_n(a,b) = \int_0^\infty dv\, v^n \sin\left(av - \frac{b}{v} \right). \quad (199)$$

$$a = \frac{z - z'}{2\hbar}, \quad b = \frac{1}{2\hbar} \int_{\phi}^{\phi'} d\chi \left[(e\mathbf{A}_\perp(\chi) - \mathbf{p}_\perp)^2 + m^2c^2 \right], \quad ab > 0 \quad (200)$$

Completeness of Dirac Volkov solutions III

$$\Gamma_0(\mathbf{p}_\perp,\phi,\phi') = 1 - \alpha_3$$

$$\Gamma_{-1}(\mathbf{p}_\perp,\phi,\phi') = -\mathrm{i}\left\{\left[2(\widehat{p}_\perp + mc) - e\left(\widehat{A}_\perp(\phi) + \widehat{A}_\perp(\phi')\right)\right]\gamma^0 + e\left(\widehat{A}_\perp(\phi) - \widehat{A}_\perp(\phi')\right)\gamma^3\right\}$$

$$\Gamma_{-2}(\mathbf{p}_\perp,\phi,\phi') = \left[\mathbf{p}_\perp^2 + m^2c^2 + e\left(\widehat{A}_\perp(\phi)\widehat{p}_\perp + \widehat{p}_\perp\widehat{A}_\perp(\phi')\right) - e^2\widehat{A}_\perp(\phi)\widehat{A}_\perp(\phi') + mc\left(e\widehat{A}_\perp(\phi') - e\widehat{A}_\perp(\phi)\right)\right](1+\alpha_3).$$

$$S_{-1}(a,b) = 0, \qquad C_0(a,b) = \pi\delta(a), \qquad C_{-2}(a,b) = \pi\delta(b). \tag{201}$$

$$\delta\left(\frac{z-z'}{2\hbar}\right) = 2\hbar\delta(z-z'), \tag{202}$$

$$\delta\left(\frac{1}{2\hbar}\int_\phi^{\phi'} d\phi\left[(e\mathbf{A}_\perp(\chi) - \mathbf{p}_\perp)^2 + m^2c^2\right]\right) = \frac{2\hbar}{(e\mathbf{A}_\perp(\phi) - \mathbf{p}_\perp)^2 + m^2c^2}\delta(z-z')$$

Completeness of Dirac Volkov solutions IV

$$ \tag{203} $$

$$\Gamma_{-2}(\mathbf{p}_\perp, \phi, \phi')\Big|_{z=z'} = \left[(e\mathbf{A}_\perp(\phi) - \mathbf{p}_\perp)^2 + m^2c^2\right](1+\alpha_3). \tag{204}$$

$$C(\mathbf{r}, \mathbf{r}', t) = \frac{1}{(2\pi\hbar)^2}\int\limits_{\mathcal{R}^2} d\mathbf{p}_\perp \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathbf{p}_\perp\cdot(\mathbf{r}_\perp - \mathbf{r}'_\perp)}\left(\frac{1}{2}(1+\alpha_3) + \frac{1}{2}(1-\alpha_3)\right)\delta(z-z') \tag{205}$$

$$C(\mathbf{r}, \mathbf{r}', t) = \delta(\mathbf{r} - \mathbf{r}') \tag{206}$$

Derivation and properties of Gordon-Volkov solutions I

The Klein-Gordon Hamiltonian for a particle of mass m and electric charge $e < 0$ in the laser field $A(\phi)$ is,

$$H(x) = -c^2\Pi_\mu\Pi^\mu + m^2c^4, \quad \Pi_\mu = \mathrm{i}\hbar\partial_\mu - eA_\mu(\phi). \tag{207}$$

and the Volkov solutions of the corresponding equation

$$H(x)\psi(x) = 0, \tag{208}$$

are

$$\psi_\pm(p; x) = \frac{1}{\sqrt{2E}(2\pi\hbar)^{3/2}} e^{\mp\frac{\mathrm{i}}{\hbar}p\cdot x \pm \frac{\mathrm{i}}{2\hbar(n\cdot p)}\int_{\phi_0}^{\phi} d\chi\left[e^2A^2(\chi)\mp 2eA(\chi)\cdot p\right]}. \tag{209}$$

The inner-product definition:

$$\begin{aligned}\langle f(x), g(x)\rangle &= \int d\mathbf{r} f^*(x)\left[\mathrm{i}\hbar\overset{\leftrightarrow}{\partial_0} - 2ceA_0(\phi)\right]g(x) \quad (210)\\ &\int d\mathbf{r}\left[f^*(x)\mathrm{i}\hbar\frac{\partial g(x)}{\partial t} - g(x)\mathrm{i}\hbar\frac{\partial f^*(x)}{\partial t} - 2ceA_0(\phi)f^*(x)g(x)\right].\end{aligned}$$

Derivation and properties of Gordon-Volkov solutions II

Orthogonality

$$\langle \psi_\pm(p_1;x), \psi_\pm(p_2;x)\rangle = \pm\delta(\mathbf{p}_1 - \mathbf{p}_2), \quad \langle \psi_+(p_1;x), \psi_-(p_2;x)\rangle = 0; \tag{211}$$

The general form of the completeness relation for a set of functions $\{\phi_i\}$ orthogonal with respect to an inner product,

$$\langle \phi_i, \phi_k\rangle = d_i \delta_{ik}, \tag{212}$$

is

$$\mathbf{I} = \sum_i \frac{1}{d_i}\phi_i \langle \phi_i, \cdot\rangle \tag{213}$$

where $\mathbf{I}$ is the unit operator. For us:

$$\mathbf{I} = \int d\mathbf{p}\psi_+(p)\langle\psi_+(p), \cdot\rangle - \int d\mathbf{p}\psi_-(p)\langle\psi_-(p), \cdot\rangle \tag{214}$$

and, using the definition of the Klein-Gordon scalar product one sees that the above relation is equivalent to the pair of relations

$$\mathcal{I}_1 \equiv \int d\mathbf{p}\left[\psi_+(p;x)\psi_+^*(p;x') - \psi_-(p;x)\psi_-^*(p;x')\right] = 0 \tag{215}$$

Derivation and properties of Gordon-Volkov solutions III

and

$$\begin{aligned} \mathcal{I}_2 &\equiv \int d\mathbf{p} \left[\psi_+(p;x) \left(i\hbar \frac{\partial \psi_+(p;x')}{\partial t} \right)^* - \psi_-(p;x) \left(i\hbar \frac{\partial \psi_-(p;x')}{\partial t} \right)^* \right. \quad (216) \\ &- \left. 2ceA_0(\phi') \left(\psi_+(p;x)\psi_+^*(p;x') - \psi_-(p,x)\psi_-^*(p;x') \right) \right] = \delta(\mathbf{r} - \mathbf{r}') ; \end{aligned}$$

Using $\mathcal{I}_1$ one simplifies $\mathcal{I}_2$

$$\mathcal{I}_2 = \int d\mathbf{p} \left[\psi_+(p;x) \left(i\hbar \frac{\partial \psi_+(p;x')}{\partial t} \right)^* - \psi_-(p;x) \left(i\hbar \frac{\partial \psi_-(p;x')}{\partial t} \right)^* \right] . \quad (217)$$

The transition amplitude

"Usual" expression

$$\mathcal{A}_{if} = -\frac{\mathrm{i}}{c\hbar}\int d^4x\psi_+^*(p_2;x)H_{\mathrm{int}}(x)\psi_+(p_1;x)\,. \tag{218}$$

Q: Is it valid for Klein-Gordon case?

The Klein-Gordon Feynman propagator

$$\begin{aligned} K(x,x') &= -\mathrm{i}\hbar\theta(t-t')\int d\mathbf{p}\psi_+(p,x)\psi_+^*(p;x') \\ &- \mathrm{i}\hbar\theta(t'-t)\int d\mathbf{p}\psi_-(p;x)\psi_-^*(p;x')\,; \end{aligned} \tag{219}$$

The action of the Feynman propagator on an arbitrary function

$$\phi(x) = \int d\mathbf{p}\, c_+(\mathbf{p})\psi_+(p;x) + \int d\mathbf{p}\, c_-(\mathbf{p})\psi_-(p;x) \equiv \phi_+(x)+\phi_-(x) \tag{220}$$

$$K(x',x)\phi(x) = \int d\mathbf{r} K(x',x)[\mathrm{i}\hbar\overleftrightarrow{\partial}_0 - 2ceA_0(\chi)]\phi(x), \tag{221}$$

The transition amplitude I

$$K(x',x)\phi(x) = -\mathrm{i}\hbar\theta(t'-t)\phi_+(x') + \mathrm{i}\hbar\theta(t-t')\phi_-(x') \qquad (222)$$

which means that $K(x',x)$ propagates forward in time any function $\phi_+(x')$ *originating* from positive energy solutions, and backward in time the functions $\phi_-(x')$ *originating* from negative energy solutions. The role played by the Feynman propagator of the Klein-Gordon equation for the charged particle in a plane-wave electromagnetic field is identical to that of the free propagator in the free particle case.

Using the properties of the Volkov solutions, one can easily see that $K(x',x)$ is also a Green function of the Klein-Gordon equation. Indeed, using the Hamiltonian (207) and the expression (219) of $K(x,x')$, one obtains by direct calculation

$$H(x)K(x,x') = -\hbar^2\delta(t-t')\delta(\mathbf{r}-\mathbf{r}') \qquad (223)$$

The particle in the laser field and an interaction

$$H = -c^2\Pi_\mu\Pi^\mu + m^2c^4 + H_{\mathrm{int}}(x) \equiv H_0 + H_{\mathrm{int}}(x) \qquad (224)$$

The transition amplitude II

We follow exactly the same steps as in the case of the free particle under the action of $H_{\rm int}$, assumed of finite duration,

$$\lim_{t\to\pm\infty} H_{\rm int}(x) = 0 \tag{225}$$

but use Volkov states instead of free plane-waves; the $\mathcal{S}$ matrix element between the initial state and a Volkov state of positive energy and momentum p_2 is

$$\mathcal{S}_{if} = \langle \psi_+(p_2; x), \psi_i(x)\rangle\Big|_{t\to\infty}. \tag{226}$$

In the previous equation $\psi_i(x)$ is the solution of the Klein-Gordon equation with the Hamiltonian (224) which evolves from the initial state, assumed a Volkov state of positive energy and momentum p_1 state at $t \to -\infty$; for the case of a finite laser pulse, $\psi_i(x)$ reduces at $t \to -\infty$ to a plane wave free solution. The solution $\psi_i(x)$ obeys the integral equation

$$\psi_i(x) = \psi_+(p_1; x) + \frac{1}{\hbar^2 c}\int d^4y K(x, y) H_{\rm int}(y)\psi_i(y) \tag{227}$$

The transition amplitude III

with $K(x, y)$ is the Green function (219) of the Klein-Gordon equation, analyzed before. In the first order perturbation theory with respect to $H_{\rm int}(x)$, $\psi_i(x)$ is approximated as

$$\psi_i(x) \approx \psi_+(p_1; x) + \frac{1}{\hbar^2 c} \int d^4y K(x, y) H_{\rm int}(y) \psi_+(p_1; y) \tag{228}$$

and the $\mathcal{S}$ matrix becomes

$$\mathcal{S}_{if} \approx \delta(\mathbf{p}_1 - \mathbf{p}_2) + \mathcal{A}_{if} \tag{229}$$

with

$$\mathcal{A}_{if} = \frac{1}{\hbar^2 c} \int d^4x' \psi_+(p_1; x') H_{\rm int}(x') \lim_{t\to\infty} \langle \psi_+(p_1; x), K(x, x') \rangle. \tag{230}$$

Using in the previous equation the expression (219) of the propagator and the properties of the Volkov solutions one obtains

$$\mathcal{A}_{if} = -\frac{\mathrm{i}}{c\hbar} \int d^4x \psi_+^*(p_2; x) H_{\rm int}(x) \psi_+(p_1; x)\,. \tag{231}$$

Applications I

Volkov states: solutions of the Dirac equation in a plane-wave field $A \equiv A(\phi)$.

$$A(\phi) = (0; \mathbf{A}(\phi)), \quad \mathbf{A}(\phi) = \mathbf{e_x} A_x(\phi). \tag{232}$$

Typical choice:

$$A_x(\phi) = A_0 f(\phi) \sin\left(\frac{\omega}{c}\phi\right) \equiv A_0 f(\phi) \sin k\phi, \quad k = \frac{\omega}{c} \tag{233}$$

$$\lim_{\phi \to \pm\infty} f(\phi) = 0, \quad f(\phi) = \sin^2\left(\frac{k\phi}{2N_t}\right), \quad k\phi \in (0, 2N_t\pi). \tag{234}$$

The free particle case:

$$|\Psi^{(0)}(\mathbf{r}, t_0)\rangle = \sum_{i=1,4} \int d\mathbf{p} c_i(\mathbf{p}) |\psi_i^{(0)}(p; \mathbf{r}, t_0)\rangle \tag{235}$$

with $|\psi_i^{(0)}(p; \mathbf{r}, t)\rangle$ solutions of the free Dirac equation Example:

$$c_i(\mathbf{p}) \sim \exp(-\sigma/2(\mathbf{p} - \mathbf{p}_0)^2), \tag{236}$$

Applications II

with the normalization condition:

$$\sum_{i=1,4}\int d\mathbf{p}|c_i(\mathbf{p})|^2 = 1 \tag{237}$$

For

$$c_1(\mathbf{p}) \equiv c(\mathbf{p}) = \left(\frac{\sigma}{\pi}\right)^{3/2}\exp(-\sigma\mathbf{p}^2); \tag{238}$$

one obtains the electron initially at rest

The time evolution:

$$|\Psi^{(0)}(\mathbf{r},t)\rangle = \int d\mathbf{p}c(\mathbf{p})|\psi_i^{(0)}(p;\mathbf{r},t)\rangle \tag{239}$$

In the presence of a laser field: superposition of Volkov states

$$|\Psi(\mathbf{r},t)\rangle = \int d\mathbf{p}c(\mathbf{p})|\psi(p;\mathbf{r},t)\rangle. \tag{240}$$

Applications III

The probability to find the particle " in a Volkov state" of momentum **p**

$$\mathcal{P}(p) = |c(\mathbf{p})|^2 \tag{241}$$

At the end of the pulse the Volkov states reduce to the free states up to a phase, i.e.

$$|\Psi(\mathbf{r}, t \to \infty)\rangle = \int d\mathbf{p} c(\mathbf{p}) e^{iF(\mathbf{p})} |\psi^{(0)}(p; \mathbf{r}, t)\rangle. \tag{242}$$

Then

$$\mathcal{P}(p) = |c(\mathbf{p})|^2 \tag{243}$$

the phase $F(\mathbf{p})$ is responsible for a translation

Proof of Floquet theorem I

The case of a time-dependent Hamiltonian

$$\hat{\mathcal{H}}(t+T) = \hat{\mathcal{H}}(t)\,. \tag{244}$$

use the notation $\omega = 2\pi/T$.

The Floquet theory: introduced by G. Floquet at the end of XIX century in Math. In Physics

- Shirley (1965)
- Sambe (1973)
- Fainsthein, Manakov si Rapoport(1978)

Proof of Floquet theorem II

For a time periodic Hamiltonian theere is a solution of TDSE with the structure

$$\mid \Psi_{\mathrm{F}}(t) \rangle = e^{-\frac{i}{\hbar} Wt} \mid \Phi_{\mathrm{F}}(t) \rangle \quad \text{with} \quad \mid \Phi_{\mathrm{F}}(t+T) \rangle = \mid \Phi_{\mathrm{F}}(t) \rangle \,. \tag{245}$$

NB: Similar to the Bloch Theorem.

Proof:

Change of variable $t \to t + T$ and use the periodicity of $\mathcal{H}$

$$i\hbar \frac{d \mid \Psi(t+T) \rangle}{dt} = \hat{\mathcal{H}}(t) \mid \Psi(t+T) \rangle \,,$$

i.e. if $\mid \Psi(t) \rangle$ is solution also $\mid \tilde{\Psi}(t) \rangle \equiv \mid \Psi(t+T) \rangle$ is solution (generally not the same)

One defines a Floquet solution as the solution for which

$$\mid \Psi_{\mathrm{F}}(t+T) \rangle = \lambda \mid \Psi_{\mathrm{F}}(t) \rangle \,, \tag{246}$$

any t, with constant λ

Properties of Floquet solutions:

Proof of Floquet theorem III

- The norm is constant in time (if the solution *can* e normalized)

$$\langle \Psi_{\rm F}(t) \mid \Psi_{\rm F}(t) \rangle = \langle \Psi_{\rm F}(t+T) \mid \Psi_{\rm F}(t+T) \rangle \,, \tag{247}$$

i.e. $\mid \lambda \mid = 1$. Then use the notation

$$\lambda = e^{-\frac{i}{\hbar}WT} \,, \qquad W = W^* \,. \tag{248}$$

- If the solution can't be normalized, use the same notation but with complex W

In general case write the solution as

$$\mid \Psi_{\rm F}(t) \rangle = e^{-\frac{i}{\hbar}Wt} \mid \Phi(t) \rangle \tag{249}$$

From the definition

$$\mid \Psi_{\rm F}(t+T) \rangle = \lambda \mid \Psi_{\rm F}(t) \rangle \,, \tag{250}$$

we get

$$\mid \Psi_{\rm F}(t+T) \rangle = e^{-\frac{i}{\hbar}WT} \mid \Psi_{\rm F}(t) \rangle \,,$$

or

$$e^{-\frac{i}{\hbar}W(t+T)} \mid \Phi(t+T) \rangle = e^{-\frac{i}{\hbar}WT} e^{-\frac{i}{\hbar}Wt} \mid \Phi(t) \rangle \,,$$

i.e. $\mid \Phi(t+T) \rangle = \mid \Phi(t) \rangle$. (QED)

Quasienergies; Floquet maps I

For a solution of the form

$$|\,\Psi_{\mathrm{F}}(t)\,\rangle = e^{-\frac{i}{\hbar}Wt}\,|\,\Phi_{\mathrm{F}}(t)\,\rangle \quad \text{with} \quad |\,\Phi_{\mathrm{F}}(t+T)\,\rangle = |\,\Phi_{\mathrm{F}}(t)\,\rangle\,. \tag{251}$$

W (in general a complex number) is named *quasienergy*.

Obs: the quasienergies are not well defined, but up to a multiple of $\hbar\omega$ *.*

$$|\,\Psi_{\mathrm{F}}(t)\,\rangle = e^{-\frac{i}{\hbar}Wt}\,|\,\Phi(t)\,\rangle = e^{-\frac{i}{\hbar}\tilde{W}t}\,|\,\tilde{\Phi}(t)\,\rangle\,, \tag{252}$$

with

$$\tilde{W} = W + N\hbar\omega\,, \qquad |\,\tilde{\Phi}(t)\,\rangle = e^{iN\omega t}\,|\,\Phi(t)\,\rangle\,, \tag{253}$$

But also $|\,\tilde{\Phi}(t)\,\rangle$ leads to a Floquet solution.

Usually all the quasienergies are brought in the same interval

Quasienergies; Floquet maps II

$E - \hbar\omega/2 \leq W < E + \hbar\omega/2$, cu E The calculation is done numerically; by expanding $|\Phi_F(t)\rangle$ in a Fourier series

$$| \Phi_F(t) \rangle = \sum_{n=-\infty}^{\infty} e^{-in\omega t} | \Phi_n \rangle , \tag{254}$$

where $| \Phi_n \rangle$ the Fourier-Floquet components. Also the Hamiltonian can be expanded

$$\hat{\mathcal{H}}(t) = \sum_{n=-\infty}^{\infty} e^{-in\omega t} \hat{H}_n , \tag{255}$$

One obtains the system of Floquet equations

$$\sum_{n'=-\infty}^{\infty} \hat{H}_{n-n'} | \Phi_{n'} \rangle = (W + n\hbar\omega) | \Phi_n \rangle , \quad n = -\infty, \dots , -1, 0, 1, \dots \infty , \tag{256}$$

For a Floquet solution

$$| \Psi_{\mathrm{F}}(t) \rangle = e^{-\frac{i}{\hbar} Wt} | \Phi_{\mathrm{F}}(t) \rangle \tag{257}$$

Quasienergies; Floquet maps III

with the notation $W = W_r - \frac{i\Gamma}{2}$ we have

$$\langle \Psi_{\rm F}(t) | \Psi_{\rm F}(t) \rangle \sim e^{-i(W-W^*)t/\hbar} \tag{258}$$

$$-i(W - W^*) = 2Im(W) \equiv -\Gamma, \tag{259}$$

then

$$\langle \Psi_{\rm F}(t) | \Psi_{\rm F}(t) \rangle \sim e^{-\Gamma t}, \quad \Gamma = -2Im(W) \tag{260}$$

Γ is the ionization rate. The solutions with negative Γ are unphysical (named ghost solutions). In the absence of the electromagnetic field the Hamiltonian (constant) still has a Floquet problem:

$$\sum_{n'=-\infty}^{\infty} \hat{H}_{n-n'} \mid \Phi_{n'} \rangle = (W + n\hbar\omega) \mid \Phi_n \rangle\,, \quad n = -\infty, \ldots, -1, 0, 1, \ldots \infty\,, \tag{261}$$

with $H_n \sim \delta_n$ the system of equations reduces to

$$\hat{H}_0 \mid \Phi_n \rangle = (W + n\hbar\omega) \mid \Phi_n \rangle\,, \quad n = -\infty, \ldots, -1, 0, 1, \ldots \infty\,, \tag{262}$$

Quasienergies; Floquet maps IV

i.e.

$$W_n = E_a + n\hbar\omega \tag{263}$$

where E_a are the atomic levels. For a given electromagnetic field (normally in the dipole approximation) with the vector potential

$$\mathbf{A}(t) = A_0 \left[\cos\zeta/2 \, \cos(\omega t + \delta_0)\, \mathbf{s}_1 + \sin\zeta/2 \, \sin(\omega t + \delta_0)\, \mathbf{s}_2 \right] , \tag{264}$$

electric field

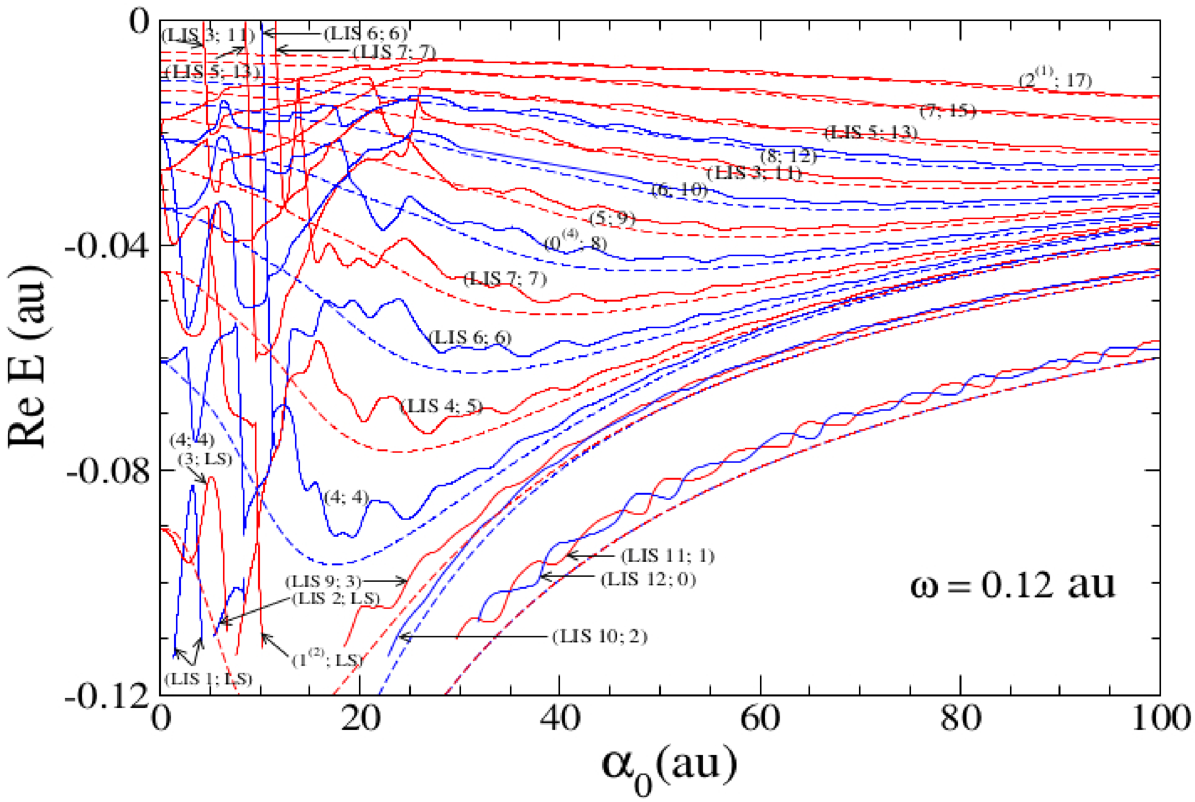
$$\mathbf{E}(t) = A_0\,\omega \left[\cos\zeta/2 \, \sin(\omega\, t + \delta_0)\, \mathbf{s}_1 - \sin\zeta/2 \, \cos(\omega\, t + \delta_0)\, \mathbf{s}_2 \right] \tag{265}$$

we define

$$\boldsymbol{\alpha}(t) \equiv \frac{e}{m_e}\frac{\mathbf{E}(t)}{\omega^2} , \quad \alpha_0 = \frac{|e|}{m_e}\frac{A_0}{\omega} \tag{266}$$

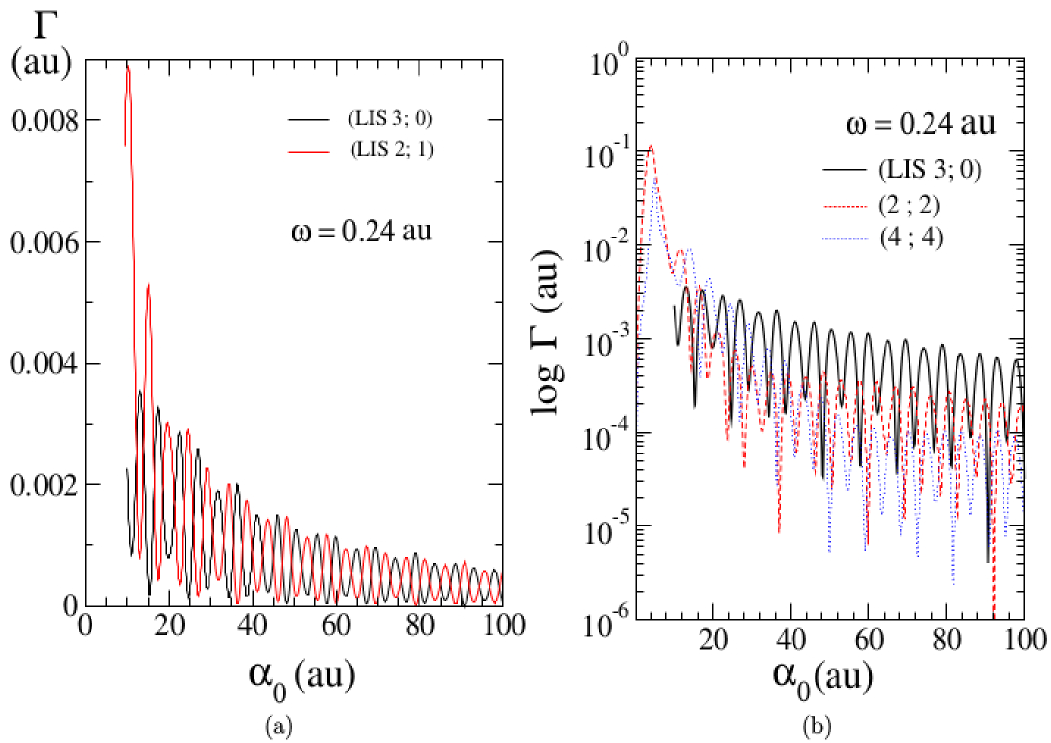
One can solve the Floquet systems of equations, and calculate the quasienergy W for different values of α (i.e. different intensities). A Floquet map: the graphical representation of W as a function of α.

Quasienergies; Floquet maps V



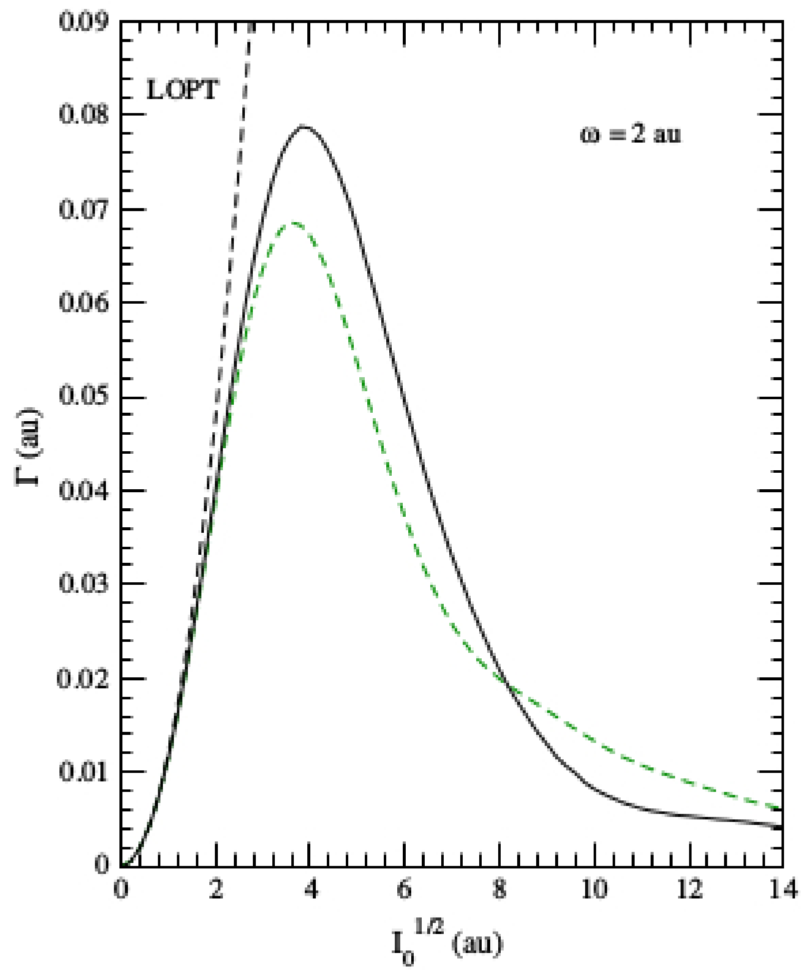
M. Stroe, PhD thesis, 2009, UB,

Quasienergies; Floquet maps VI



M. Stroe, PhD thesis, 2009, UB,

Quasienergies; Floquet maps VII



M. Boca, H. G. Muller, M. Gavrila, 2004

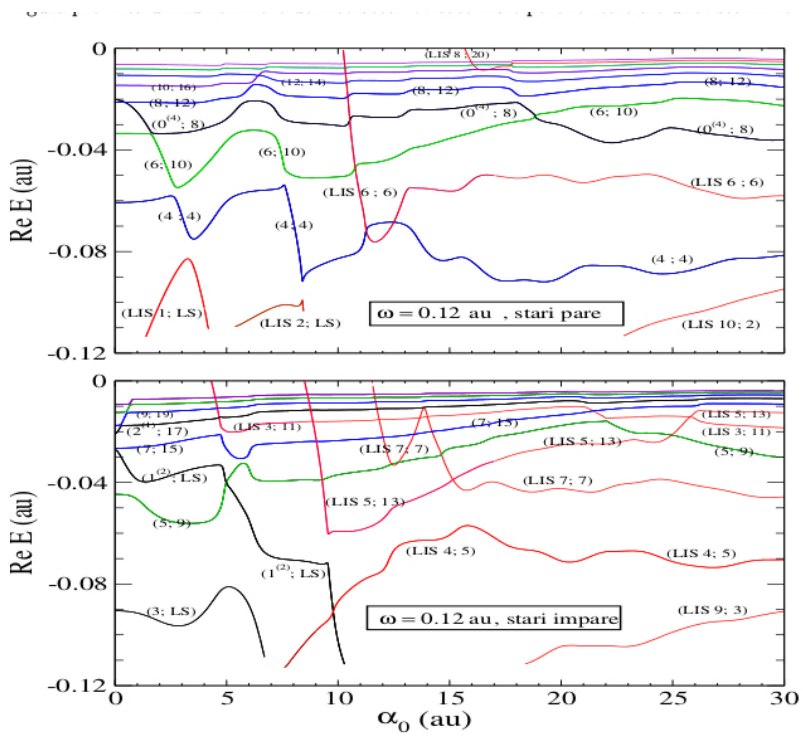
The atomic stabilization I

- In the presence of the external electromagnetic field the energy levels are shifted and they aquire a width, with the significance of ionization rate.
- Atomic stabilization: the tendency of a tom to become stable against ionization for large field intensity. Explained by the behaviour of the imaginary parts of the Floquet quasienergies. They decrease when α_0 increases; the stabilization is more efficient at high frequencies.
- **this is named stationary stabilization**, i.e. the stabilization in the monochromatic regime.

In the realistic case of a finite laser pulse, the state of the system evolves in time; the evolution is either along a Floquet state (adibatic), or along a path consisting of diffrent states (diabatic).

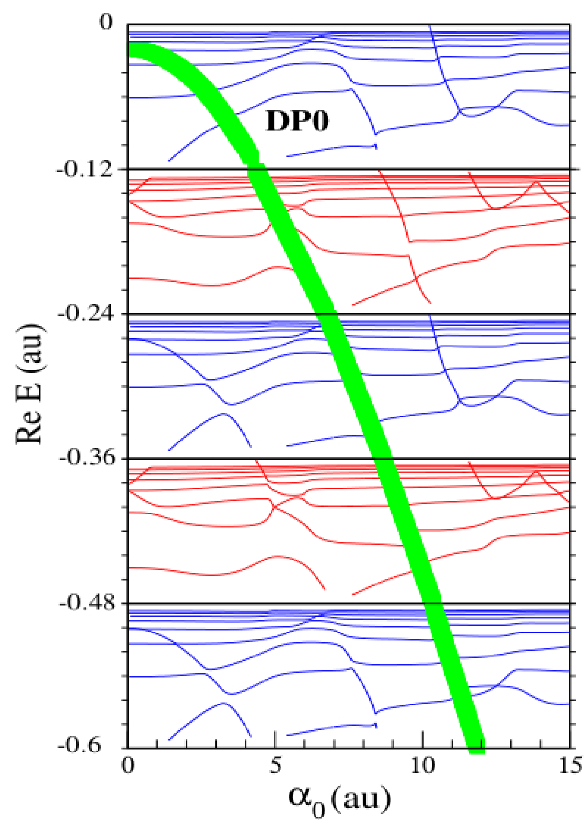
The case of diabatic evolution: one defines diabatic paths.

The atomic stabilization II



M. Stroe, PhD thesis, 2009, UB,

The atomic stabilization III



The atomic stabilization IV

M. Stroe, PhD thesis, 2009, UB,

Under the effect of a pulse $E_0 = E_0(t)$, we have:

$$P^{ion} = 1 - e^{-\int dt \Gamma(E_0(t))} \tag{267}$$

Dynamic stabilization: the decrese of the total ionization proability at the end of a laser pulse, as a function of the pulse intensity.

Effect of frequency and laser pulse shape. If more than one path is possible then we have effects of the path (controled also by the pulse shape)

The atomic stabilization V

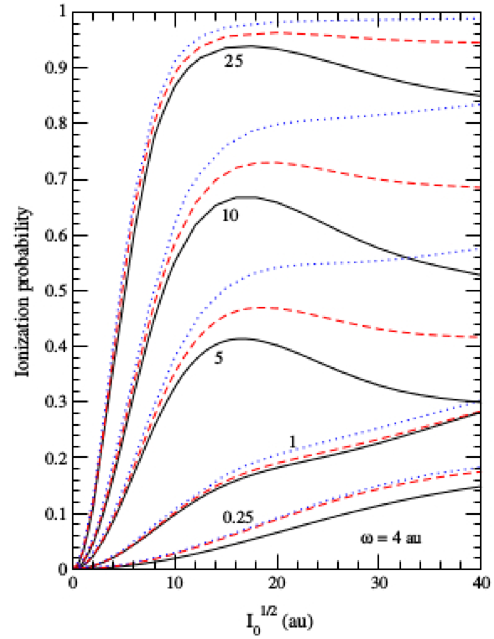


Figure 6. The same as for figure 4, except that $\omega = 4$ au.

M. Boca, H. G. Muller, M. Gavrila,

Floquet theory in the oscillating frame I

After an unitary transformation we obtain the Schrodinger equation in the oscillating frame

$$\left[\frac{\mathbf{P}^2}{2m_\mathrm{e}} + V\left(\mathbf{r} + \boldsymbol{\alpha}(t)\right)\right]\Psi_{\mathrm{KH}}(\mathbf{r},t) = i\hbar\frac{\partial\Psi_{\mathrm{KH}}(\mathbf{r},t)}{\partial t}\,, \tag{268}$$

α is the classical trajectory of the free electron in the field. Write the electromagnetic field as (dipole approximation)

$$\mathbf{A}(t) = A_0\left[\cos\zeta/2\,\cos(\omega t+\delta_0)\,\mathbf{s}_1 + \sin\zeta/2\,\sin(\omega t+\delta_0)\,\mathbf{s}_2\right]\,, \tag{269}$$

$$\mathbf{E}(t) = A_0\,\omega\left[\cos\zeta/2\,\sin(\omega\,t+\delta_0)\,\mathbf{s}_1 - \sin\zeta/2\,\cos(\omega\,t+\delta_0)\,\mathbf{s}_2\right] \tag{270}$$

then the trajectory

$$\boldsymbol{\alpha}(t) \equiv \frac{e}{m_\mathrm{e}}\frac{\mathbf{E}(t)}{\omega^2}\,. \tag{271}$$

The Floquet theory in the KH (oscilating) frame: the Floquet theory applied for the TDSE in the KH frame

Floquet theory in the oscillating frame II

Solution:

$$\Psi_{\mathrm{KH}}(\mathbf{r}, t) = e^{-\frac{i}{\hbar} W_{\mathrm{KH}} t} \Phi_{\mathrm{KH}}(\mathbf{r}, t) \,, \tag{272}$$

with $\Phi_{\mathrm{KH}}(\mathbf{r}, t)$ periodic

The components Fourier Floquet $\Phi_n^{\mathrm{KH}}(\mathbf{r})$, obey the equations

$$\left(W + n\hbar\omega - \frac{\mathbf{P}^2}{2m_e} \right) \Phi_n^{\mathrm{KH}}(\mathbf{r}) = \sum_{n'=-\infty}^{\infty} V_{n-n'}(\mathbf{r}) \, \Phi_{n'}^{\mathrm{KH}}(\mathbf{r}) \,, \tag{273}$$

$$V_n(\mathbf{r}) = \frac{1}{T} \int_0^T e^{in\omega t} \, V(\mathbf{r} + \boldsymbol{\alpha}(t)) \, dt \,. \tag{274}$$

obs: the general form was

$$\sum_{n'=-\infty}^{\infty} \hat{H}_{n-n'} \mid \Phi_{n'} \rangle = (W + n\hbar\omega) \mid \Phi_n \rangle \,, \quad n = -\infty, \ldots, -1, 0, 1, \ldots \infty \,, \tag{275}$$

Important property: at large distances the equations are not coupled, i.e. the assimptotic conditions are easy to be imposed

Floquet theory in the oscillating frame III

Large distances:

$$-\frac{\hbar^2}{2m_\mathrm{e}}\Delta\Phi_n^{\mathrm{KH}} = (W + n\hbar\omega)\,\Phi_n^{\mathrm{KH}}\,, \tag{276}$$

define

$$\frac{\hbar^2 k_n^2}{2m_\mathrm{e}} \equiv W + n\hbar\omega\,, \tag{277}$$

for real W: define:

- closed channels: $W + n\hbar\omega < 0$: $k_n \in R$

$$\Phi_n \sim \frac{1}{r}e^{\pm i k_n r} \tag{278}$$

- open channels: $W + n\hbar\omega > 0$: $k_n = i\kappa_n$

$$\Phi_n \sim \frac{1}{r}e^{-\kappa_n r} \tag{279}$$

Floquet theory in the oscillating frame IV

for complex W: ionization conditions: define:

$$k_n = \mathrm{Re}\, k_n + i\,\mathrm{Im}\, k_n \,, \tag{280}$$

$$\frac{\hbar^2}{2m_\mathrm{e}}\left[(\mathrm{Re}\, k_n)^2 - (\mathrm{Im}\, k_n)^2\right] = \mathrm{Re}\, W + n\hbar\omega \,, \qquad \frac{\hbar^2}{m_\mathrm{e}}\mathrm{Re}\, k_n \cdot \mathrm{Im}\, k_n = \mathrm{Im}\, W \,. \tag{281}$$

- $\mathrm{Re}\, W + n\hbar\omega > 0$ for open channels
- $\mathrm{Re}\, W + n\hbar\omega < 0$ for clased channels

$$\begin{array}{rcl} \Phi_n^{\mathrm{KH}}(\mathbf{r}) & \rightarrow & f_n(\hat{\mathbf{r}})\dfrac{e^{ik_n r}}{r} \,, \qquad \mathrm{Re}\, k_n > 0 \quad \text{open} \,, \\ r & \rightarrow & \infty \qquad\qquad\quad \mathrm{Im}\, k_n > 0 \,, \quad \text{closed} \,. \end{array} \tag{282}$$

Floquet theory in the oscillating frame V

i.e. divergent wave in the open channels and decreasing in the closed channels.
Redefinition of choice of k_n

$$\mathrm{Re}\, k_n > 0\,, \quad \mathrm{open} \qquad \mathrm{Im}\, k_n > 0\,, \quad \text{closed} \tag{283}$$

Phisycal interpretation: related to the the density current

$$\mathcal{J}(\mathbf{r},t) \equiv \frac{\hbar}{2im_{\mathrm{e}}}\left(\Psi_{\mathrm{F}}^{*}\boldsymbol{\nabla}\Psi_{\mathrm{F}} - \Psi_{\mathrm{F}}\boldsymbol{\nabla}\Psi_{\mathrm{F}}^{*}\right) = \tag{284}$$

$$\frac{\hbar}{2im_{\mathrm{e}}}\left(\Phi^{*}\boldsymbol{\nabla}\Phi - \Phi\boldsymbol{\nabla}\Phi^{*}\right)\exp\left(\frac{2}{\hbar}\mathrm{Im}\, Wt\right)\,. \tag{285}$$

if

$$\frac{2}{\hbar}\mathrm{Im}\, W\, T \ll 1 \qquad \mathrm{sau} \qquad \frac{\mathrm{Im}\, W}{\hbar\omega} \ll 1/4\pi\,, \tag{286}$$

approximate the time average as

$$\langle \mathcal{J} \rangle \equiv \frac{1}{T}\int_0^T \mathcal{J}\, dt \approx \frac{\hbar}{2im_{\mathrm{e}}}\sum_{n=-\infty}^{\infty}\left(\Phi_n^{*}\boldsymbol{\nabla}\Phi_n - \Phi_n\boldsymbol{\nabla}\Phi_n^{*}\right)\exp\left(\frac{2\mathrm{Im}\, W}{\hbar}\, t\right)\,. \tag{287}$$

Floquet theory in the oscillating frame VI

At large distances, (by direct calculation)

$$\langle \mathcal{J}^{\mathrm{as}} \rangle \approx \frac{\hbar}{m_{\mathrm{e}}} \hat{\mathbf{r}} \sum_n \mathrm{Re}\, k_n \frac{|\, f_n \,|^2}{r^2} \exp(-2\, \mathrm{Im}\, k_n\, r) \exp\left(\frac{2\mathrm{Im}\, W}{\hbar}\, t\right) \quad , \qquad r \to \infty \,. \tag{288}$$

i.e. only open channels contribute Also it is interesting the density probaiblity

$$\mathcal{P}(\mathbf{r}, t) \equiv |\, \Psi_{\mathrm{F}}^{\mathrm{ion}}(\mathbf{r}, t)\, |^2 = |\, \Phi(\mathbf{r})\, |^2 \exp\left(\frac{2\mathrm{Im}\, W}{\hbar}\, t\right) . \tag{289}$$

and its time average

$$\langle \mathcal{P} \rangle \equiv \frac{1}{T} \int_0^T \mathcal{P} dt \approx \sum_{n=-\infty}^{\infty} |\, \Phi_n \,|^2 \exp\left(\frac{2\mathrm{Im}\, W}{\hbar} t\right) . \tag{290}$$

The ionization rate calculated from the assymptotic curent

$$dR = \frac{\langle \mathcal{J}^{as} \cdot d\mathbf{S}_{\mathcal{R}} \rangle}{\int_V \langle \mathcal{P} \rangle d\, \mathbf{r}} \,, \qquad \mathcal{R} \to \infty \,. \tag{291}$$

Floquet theory in the oscillating frame VII

is a sum of contributions from the open channels

$$\frac{dR}{d\Omega} = \sum_{n \geq n_0}^{\infty} \frac{dR_n}{d\Omega}, \tag{292}$$

$$\frac{dR_n}{d\Omega} = \frac{\hbar}{m_e} \lim_{\mathcal{R} \to \infty} \frac{\mathrm{Re}\, k_n \mid f_n \mid^2 e^{-2\,\mathrm{Im}\, k_n\, \mathcal{R}}}{\int_V \sum_{n'} \mid \Phi_{n'} \mid^2 d\mathbf{r}}, \qquad n \geq n_0\,. \tag{293}$$

$\mathcal{R}_n$: the ionization rate in the process in which n photons are absorbed Obs: applied even if Floquet solutions are not normalizable One can prove that:

$$Im(W) = -\frac{\Gamma}{2} \tag{294}$$

$$\Gamma = \sum_{open} \Gamma_n = \sum_{open} \int d\Omega \frac{d\Gamma_n}{d\Omega} > 0 \tag{295}$$

The imaginary part of the quasienergy is $-1/2\times$ the total ionization rate and is negative.

High frequency Floquet theory I

Write the Floquet system of equations

$$\left(W + n\hbar\omega - \frac{\mathbf{P}^2}{2m_e} - V_0(\mathbf{r}) \right) \Phi_n^{\mathrm{KH}}(\mathbf{r}) = \sum_{n \neq n'} V_{n-n'}(\mathbf{r}) \Phi_{n'}(\mathbf{r}) \,. \tag{296}$$

and define

$$H_{\mathrm{d}} \equiv \frac{\mathbf{P}^2}{2m_e} + V_0(\mathbf{r}) \,. \tag{297}$$

H_0 the hamiltonian of a fictitious system with the potential

$$V_0(\mathbf{r}) = \frac{1}{T} \int_0^T V(\mathbf{r} + \boldsymbol{\alpha}(t))\, dt \,. \tag{298}$$

HF limit (i.e. high frequencies and fixed α (high intensity))

The frequency does not appear in RHS, then in the HF limit

$$\Phi_n^{\mathrm{HF}} = 0 \,, \qquad n \neq 0 \,, \qquad \Phi_0^{\mathrm{HF}}(\mathbf{r}) \neq 0 \,, \tag{299}$$

High frequency Floquet theory II

$$\left[W^{\rm HF} - \frac{\mathbf{P}^2}{2m_e} - V_0(\mathbf{r}) \right] \Phi_0^{\rm HF}(\mathbf{r}) = 0 \,, \qquad (300)$$

In the high frequency limit the Floquet system of equation reduces to the equation of structure, which has real eigenvalues W^{HF}. In the HFFT there is no ionization.

Validity criterion: $\hbar\omega \gg |E_0|$,, where E_0 is the ground state energy if the system.

Corrections to the high frequency limit: one can calculate analitically the first order correction to the quasienergy

By iterating the Floquet system of equations

$$\left(W^{\rm HF} + n\hbar\omega - \frac{P^2}{2m_e} - V_0(\mathbf{r}) \right) \Phi_n^{(1)}(\mathbf{r}) = V_n(\mathbf{r}) \Phi_0^{(1)}(\mathbf{r}) \,, \qquad n \neq 0 \,. \qquad (301)$$

one can prove that

$$\Gamma^{(1)} = \frac{m_e}{4\pi^2\hbar^3} \sum_{n=n_0}^{\infty} k_n \int \mid \langle \mathbf{k}_n - \mid V_n \mid \Phi_0^{\rm HF} \rangle \mid^2 d\,\Omega_{\mathbf{k}_n} \,, \qquad (302)$$

$\mathbf{k}_n \equiv k_n \hat{\mathbf{r}} \,, \ \langle \mathbf{r} \mid \mathbf{k}_n - \rangle = u^{(-)}(\mathbf{k}_n; \mathbf{r})$

Atomic stabilization: the validity criterion of the HFFT is replaced by:

High frequency Floquet theory III

- Validity criterion: $\hbar\omega \gg |W_0|$,, where W_0 is the ground state of the structure equation.
- But, $|W_0|$ decreases with α_0 (i.e. with intensity).
- conclusion: at large enough intensity the HFFT is **allways** valid

Results for Hidrogen atom: The dressed potential V_0 becomes singular along the path of the trajectory $\alpha(t)$ (M. Gavrila): in the case of linear polarization: log singularity along the trajectory and $1/x^{-1/2}$ at the end points; for circular polarization log singularity along the circle.

High frequency Floquet theory IV

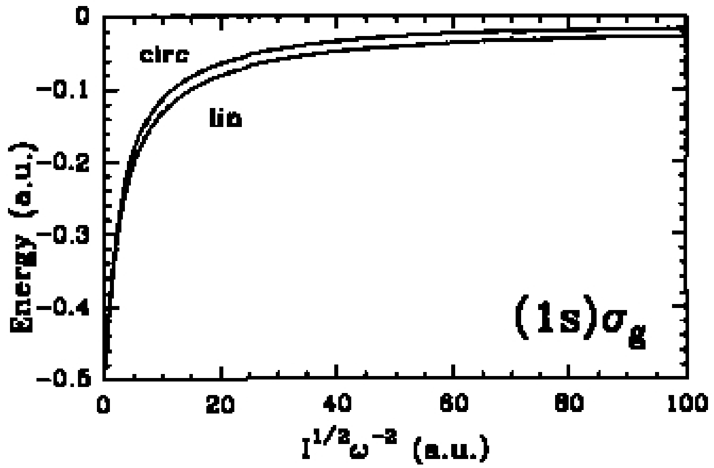


FIG. 1. High-frequency limit energy of the hydrogen ground state $(1s)\sigma_g$, for linear and circular polarizations, as function of $I^{1/2}\omega^{-2}$. For linear polarization $\alpha_0^l = I^{1/2}\omega^{-2}$, and for circular polarization $\alpha_0^c = (I/2)^{1/2}\omega^{-2}$, see Eq. (128). Based on results by Pont *et al.* (1988) and Vos and Gavrila (unpublished).

High frequency Floquet theory V

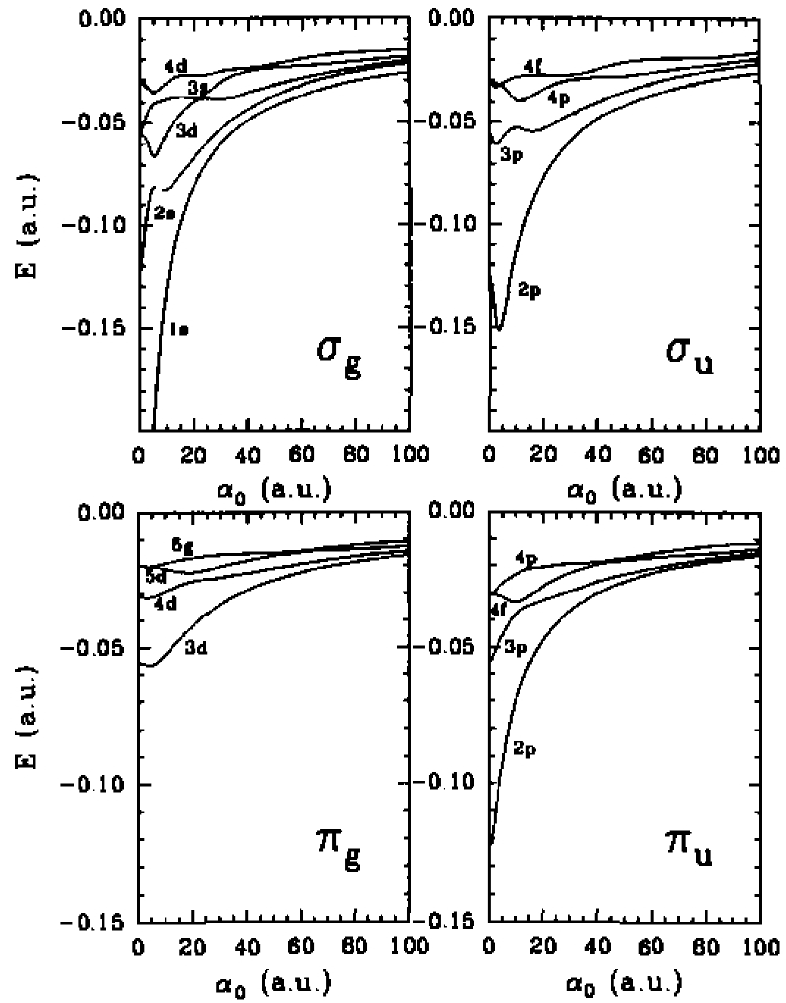


FIG. 2. High-frequency limit energies of the hydrogen first few states belonging to the symmetry manifolds σ_g, σ_u, π_g, π_u, for linear polarization, as functions of α_0'. From Pont *et al.* (1990), Fig. 1.

High frequency Floquet theory VI

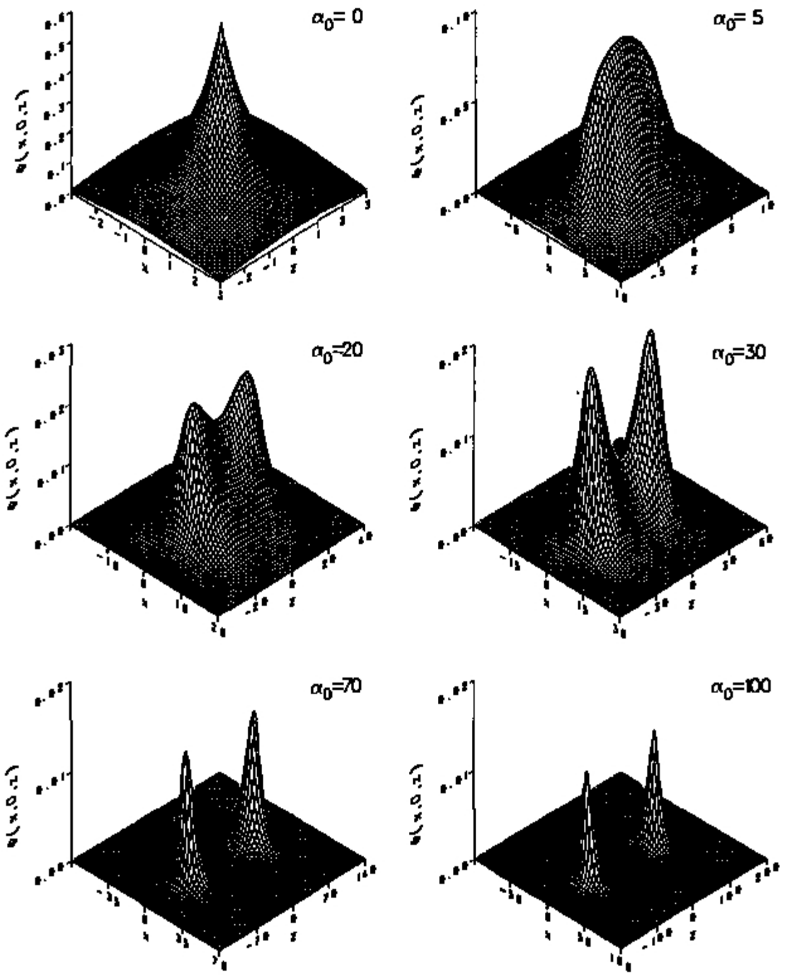


FIG. 4. High-frequency limit energy eigenfunction of the normalized ground state $(1s)\sigma_g$ of hydrogen in the oscillating frame of reference for increasing α_0'; case of linear polarization. $\phi(x, 0, z)$ is the wave function in the xz plane, where the z axis is chosen along the axis of symmetry of the dressed potential, Eq. (129), and the x axis is arbitrary. Atomic units are used. There is a change of length scale in the figures as α_0' increases. Note that *dichotomy* sets in between $\alpha_0' = 30$ and 70. From Pont *et al.* (1988), Fig. 1.

Perturbation theory in the interaction picture I

TDSE in the Schrodinge picture:

$$i\hbar\frac{\partial \Psi}{\partial t} = H_0\Psi + V(t)\Psi \tag{303}$$

Assume H_0 time-independent with the eigenvalue problem

$$H_0|j\rangle = \epsilon_j|j\rangle \equiv \hbar\omega_j|j\rangle \tag{304}$$

and define the unitary transformation

$$\Psi = e^{-\frac{i}{\hbar}H_0 t}\psi \tag{305}$$

The the new state vector ψ obeys the equation

$$i\hbar\frac{\partial \psi}{\partial t} = \hat{V}(t)\psi \tag{306}$$

with

$$\hat{V}(t) = e^{\frac{i}{\hbar}H_0 t}V(t)e^{-\frac{i}{\hbar}H_0 t} \tag{307}$$

Perturbation theory in the interaction picture II

The implicit equation for the solution ψ

$$\psi(t) = |i\rangle - \frac{i}{\hbar} \int_{-\infty}^{t} dt' \hat{V}(t') \psi(t') \tag{308}$$

and the transition amplitude

$$A_{if} = \langle f|\hat{A}|i\rangle \tag{309}$$

with

$$\hat{A} = I + \sum_{N\geq 1} \hat{A}^{(N)} \tag{310}$$

$$\hat{A}^{(N)}(t) = \frac{(-i)^N}{\hbar^N} \int_{-\infty}^{t} dt_1 \ldots \int_{-\infty}^{t_{N-1}} dt_N \hat{V}(t_1) \ldots \hat{V}(t_N) \tag{311}$$

The monochromatic case I

Consider the monochromatic case in the dipole approximation and length gauge. Then

$$\hat{V}(t) = e^{\frac{i}{\hbar}H_0 t}\mathbf{E}(t)\cdot\mathbf{D}e^{-\frac{i}{\hbar}H_0 t} \tag{312}$$

with

$$\mathbf{E}(t) = \frac{1}{2}\mathbf{E}e^{i\omega t} + \frac{1}{2}\mathbf{E}^* e^{-i\omega t} \tag{313}$$

The first order result at $t \to \infty$

$$\langle f|A^{(1)}|i\rangle = -\frac{i}{2\hbar}\int_{-\infty}^{\infty} dt_1 \left[e^{-i(\omega_i-\omega_f-\omega)}E\langle f|D|i\rangle + e^{-i(\omega_i-\omega_f+\omega)}E^*\langle f|D|i\rangle\right] \tag{314}$$

NB: consider only the component of D along E Result

$$\langle f|A^{(1)}|i\rangle = -\frac{\pi i}{\hbar}\left[\delta(\omega_i-\omega_f-\omega)E\langle f|D|i\rangle + \delta(\omega_i-\omega_f+\omega)E^*\langle f|D|i\rangle\right] \tag{315}$$

two terms, for abs/emission.

The monochromatic case II

The second order term

$$\langle f|A^{(2)}|i\rangle = \frac{-2\pi i}{\hbar^2}\left[\delta(\omega_i - \omega_f + 2\omega)\, T_1 + \delta(\omega_i - \omega_f)\, T_2 + \delta(\omega_i - \omega_f - 2\omega)\, T_3\right] \tag{316}$$

with

$$T_1 = \sum_j \frac{(E/2)\langle f|D|j\rangle(E/2)\langle j|D|i\rangle}{\omega_i + \omega - \omega_j} \tag{317}$$

$$\begin{aligned} T_2 &= \sum_j \frac{(E^*/2)\langle f|D|j\rangle(E/2)\langle j|D|i\rangle}{\omega_i + \omega - \omega_j} + \\ &+ \sum_j \frac{(E/2)\langle f|D|j\rangle(E^*/2)\langle j|D|i\rangle}{\omega_i - \omega - \omega_j} + \\ T_3 &= \sum_j \frac{(E^*/2)\langle f|D|j\rangle(E^*/2)\langle j|D|i\rangle}{\omega_i - \omega - \omega_j} \end{aligned} \tag{318}$$

The monochromatic case III

Compact expression if we define

$$G(\omega) = \sum_j \frac{|j\rangle\langle j|}{\omega - \omega_j} \tag{319}$$

Then

$$T_1 = (E/2)^2 \langle f|DG(\omega_i + \omega)D|i\rangle \tag{320}$$

$$T_2 = (|E|/2)^2 \langle f|DG(\omega_i + \omega)D|i\rangle + (|E|/2)^2 \langle f|DG(\omega_i - \omega)D|i\rangle$$

$$T_3 = (E^*/2)^2 \langle f|DG(\omega_i - \omega)D|i\rangle \tag{321}$$

$$\tag{322}$$

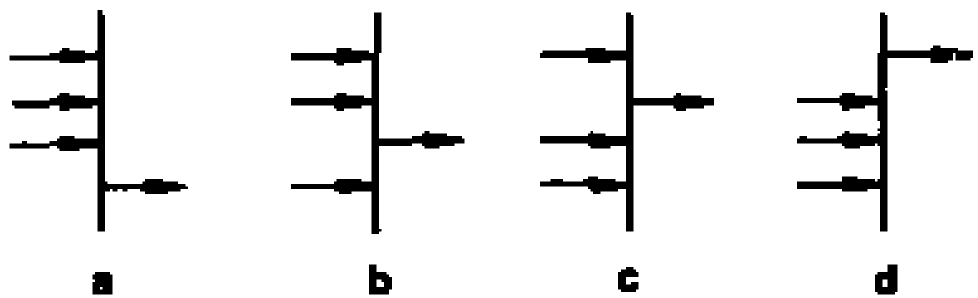
The three terms correspond to

- T_1: two photon absorbtion
- T_3: two photon emission
- T_2: one photon absorbtion+ one photon emission

The diagramatic method for monochromatic case I

Represent by arrows the emitted/absorbed photons; a N order diagram: a diagram with N arrows. It corresponds to Δn net photon exchanged ($|\Delta n| < N$). More than one distinct diagrams for given N and Δn

example: $N = 4$, $\Delta n = -2$; 4 distinct diagrams



Rules for writing a diagram

- The point at which an arrow meets the line is to be called a "vertex."

The diagramatic method for monochromatic case II

- The "net frequency" $\Delta\Omega$ in any part (or whole) of a diagram is defined as (sum of frequencies associated with the outgoing arrows)-(sum of frequencies associated with the ingoing arrows), in the considered part or whole of the diagram in question.
- The "vertex strength" is denoted by V. In our example $V = (E/2)D$
- The unperturbed "propagator" is defined by

$$G(\omega - \Delta\Omega) = \sum_j \frac{|j\rangle\langle j|}{\omega - \Delta\Omega - \omega_j} \tag{323}$$

- Read the diagram from below upward and assign a factor V at a vertex if the arrow at the vertex is ingoing, or a factor V^* if the arrow is outgoing.
- Between any two neighboring vertices assign a propagator $G(\omega_i - \Delta\Omega)$, where the "net frequency" $\Delta\Omega$ is obtained from the entire portion of the diagram below the position of the propagator.
- Multiply the vertex factors and propagator factors from right to left in the sequence of their occurrence from the bottom of the diagram upward.

The diagramatic method for monochromatic case III

- To obtain the transition amplitudes, calculate the matrix element of the above expression between $\langle f|$ and $|i\rangle$ and include the overall factor

$$-2\pi i \frac{1}{\hbar^N} \delta(\omega_i - \Delta\Omega_0 - \omega_f) \tag{324}$$

  with N: the number of vertices and $\Delta\Omega_0$ the net difference of frequencies.
- The method is valid also for fields with more colors

Example 1: two photon absorption / second order process

According to the above rules the transition amplitude becomes

$$\mathcal{A}_{if}^{(2)} = -2\pi i \frac{1}{\hbar^2} \delta(\omega_i - \omega_f + 2\omega) \langle f| \frac{E}{2} D G(\omega_i + \omega) \frac{E}{2} D |i\rangle \tag{325}$$

Example 2: two photon absorption / fourth order process

The diagramatic method for monochromatic case IV

$$\mathcal{A}_{if}^{(4)} = -2\pi i \frac{1}{\hbar^4}\delta(\omega_i - \omega_f + 2\omega) \times \tag{326}$$

$$\Big[\langle f|\frac{E}{2}DG(\omega_i+\omega)\frac{E}{2}DG(\omega_i)\frac{E}{2}DG(\omega_i-\omega)\frac{E^*}{2}D|i\rangle$$

$$\langle f|\frac{E}{2}DG(\omega_i+\omega)\frac{E}{2}DG(\omega_i)\frac{E^*}{2}DG(\omega_i+\omega)\frac{E^*}{2}D|i\rangle$$

$$\langle f|\frac{E}{2}DG(\omega_i+\omega)\frac{E^*}{2}DG(\omega_i+2\omega)\frac{E}{2}DG(\omega_i+\omega)\frac{E^*}{2}D|i\rangle$$

$$\langle f|\frac{E^*}{2}DG(\omega_i+3\omega)\frac{E}{2}DG(\omega_i+2\omega)\frac{E}{2}DG(\omega_i+\omega)\frac{E}{2}D|i\rangle \Big]$$

The transition rate and the generalized cross section

The transition probability is written as the modulus square of the transition amplitude, i.e.

$$\mathcal{P}_{if}^{(n)} = \left| -2\pi i \frac{1}{\hbar^2}\delta(\omega_i - \omega_f - (n)\omega)\langle f|\ldots|i\rangle \right|^2 \tag{327}$$

The diagramatic method for monochromatic case V

with the identity

$$\delta^2(\Omega) = \frac{T}{2\pi}\delta(\Omega) \tag{328}$$

One defines the finite transition rate

$$\mathcal{W}_{if}^{(n)} = (2\pi)\left(\frac{2\pi\alpha F\omega}{e^2}\right)^n \left|T_{if}^{(n)}\right|^2 \delta(\omega_i - \omega_f - (n)\omega) \tag{329}$$

and

$$T_{if} = \langle f|DG(\dots)D\dots|i\rangle, \quad F = \left(4\pi\epsilon_0\frac{E^2}{2}\right)\frac{1}{\hbar\omega}\frac{1}{c} \tag{330}$$

Consider two cases

- Transition to a "discrete" final state with a line-shape function

$$S(\omega) = \frac{1}{\pi}\frac{1/(2t_f)}{(\omega_i - \omega_f - (n)\omega)^2 + 1/(2t_f)^2} \tag{331}$$

The diagramatic method for monochromatic case VI

- transition to a continuum state with the density

$$\rho(\epsilon_f) = \frac{mk_f}{\hbar^2}\frac{V}{(2\pi)^3}, \qquad \frac{\hbar^2 k_f^2}{2m} = -\hbar\omega_i + (n)\hbar\omega \tag{332}$$

Integrate over energies leads to eliminatin of δ

$$d\mathcal{W}_{if}^{(n)} = (2\pi)\left(\frac{2\pi\alpha F\omega}{e^2}\right)^n \left|T_{if}^{(n)}\right|^2 S(\omega)d(\ldots) \tag{333}$$

with $(\ldots)$ other quantum numbers or

$$d\mathcal{W}_{if}^{(n)} = (2\pi)\left(\frac{2\pi\alpha F\omega}{e^2}\right)^n \left|T_{if}^{(n)}\right|^2 \hbar\rho(\epsilon_f)d\Omega \tag{334}$$

with Ω: the ionized electron direction. The total ionization rate

$$\mathcal{W}_{if}^{(n)} = \int d(\ldots)d\mathcal{W}_{if}^{(n)} \tag{335}$$

The diagramatic method for monochromatic case VII

and the generalized cross section: ratio between the rate and the incoming photon flux F

$$\sigma_{if}^{(n)} = \frac{\mathcal{W}_{if}^{(n)}}{F} \tag{336}$$

Possible alternative definition

$$\sigma_{if}^{(n)} = \frac{\mathcal{W}_{if}^{(n)}}{F^n} \tag{337}$$

with wrong dimensions The range of aplicability:

The probability ratio $p^{(n+2)}/p^{(n)}$ can be approximated as

$$\mathcal{R} = p^{(n+2)}/p^{(n)} \approx \left(\frac{2\pi\alpha F\omega}{e^2}\right)\left(\frac{ea_0}{\omega}\right)^2 \tag{338}$$

NB *it comes from*

$$\left|\mathcal{A}_{if}^{(n)} + \mathcal{A}_{if}^{(n+2)}\right|^2 \to \left|\mathcal{A}_{if}^{(n)}\right|^2 + 2Re\left[(\mathcal{A}_{if}^{(n)})^* \mathcal{A}_{if}^{(n+2)}\right] \tag{339}$$

The diagramatic method for monochromatic case VIII

in the optical domain

$$\mathcal{R} \sim \frac{I}{I_a}\left(\frac{\omega_a}{\omega}\right)^2 \tag{340}$$

($I_a = 3.51 \times 10^{16}$ W/cm^2, $\hbar\omega_a = 27.2$ eV). One obtains the domain for "normal" case ($\omega \sim 1$ eV) as $I < 10^{12}$W/cm^2

Renormalization of perturbation theory I

The goal: to remove the singularities which arise nonrelatd to phisycal resonant processes.

The origin of singuralities: let us calculate e.g the contibution of a third order diagram for three photon absorbtion: It has the form

$$\mathcal{A}_{if}^{(3)} = -2\pi i \frac{1}{\hbar^2}\delta(\omega_i - \omega_f + 3\omega)\langle f|\frac{E}{2}DG(\omega_i+\omega)\frac{E}{2}DG(\omega_i+2\omega)\frac{E}{2}D|i\rangle \tag{341}$$

$$\mathcal{A}_{if}^{(3)} \sim \sum_{k_1 k_2} \frac{\langle f|D|k_1\rangle\langle k_1|D|k_2\rangle\langle k_2|D|i\rangle}{(\omega_i+2\omega-\omega_{k_2})(\omega_i+\omega-\omega_{k_1})}\delta(\omega_i+3\omega-\omega_f) \tag{342}$$

Singularity: appears if one of the denominators vanishes, e.g. if there is a state r such that

$$\omega_i + 2\omega = \omega_r, \quad \omega_i + 2\omega = \omega_r \tag{343}$$

In general, an intermediate resonance is said to occur when one or more of the denominators vanish, which can happen if

$$\omega_r = \omega_i + n\omega \tag{344}$$

Renormalization of perturbation theory II

Such resonances are of physical interest, they describe a real enhancement of the transition probability. Another example: the third order process for one photon absorbtion contains a term of the form

$$\mathcal{A}_{if}^{(3)} \sim \sum_{k_1 k_2} \frac{\langle f|D|k_1\rangle\langle k_1|D|k_2\rangle\langle k_2|D|i\rangle}{(\omega_i - \omega_{k_2})(\omega_i + \omega - \omega_{k_1})} \tag{345}$$

It is singular if $k_2 = i$ and if $k_1 = f$; these are unphysical singularities.

For **renormalization**: Equivalent formulation of the perturbation theory.

Consider the TDSE in the Schrodinger picture, with the quantized field. Then the full Hamiltonian is time indepedent, i.e. one can write

$$\psi(t) = e^{-\frac{i}{\hbar}Ht}\phi_i \tag{346}$$

with ϕ_i the initial state (at $t \to -\infty$). Equivalent form of the propagator

$$K(t, -\infty) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE G^{+}(E) e^{-\frac{i}{\hbar}Et} \tag{347}$$

Renormalization of perturbation theory III

$$G^+(E) = \frac{1}{E - H + i\epsilon} \quad (348)$$

and

$$\psi(t) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE G^+(E) e^{-\frac{i}{\hbar}Et}\phi_i \quad (349)$$

The exact expression of the transition amplitude at the moment t

$$\mathcal{A}_{if}(t) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE \langle f|G^+(E)|i\rangle e^{-\frac{i}{\hbar}(E-E_f)t} \quad (350)$$

One can prove that

$$G^+(E) = G_0^+(E) + G_0^+(E)VG_0^+(E) + G_0^+(E)VG_0^+(E)VG_0^+(E) + \dots \quad (351)$$

with G_0 the Green function of the unperturbed system. One defines the transition operator

$$T(E) = V + VG_0^+(E)V + VG_0^+(E)VG_0^+(E)V + \dots \quad (352)$$

Renormalization of perturbation theory IV

and we can write

$$G^+(E) = G_0^+(E) + G_0^+(E)TG_0^+(E) \tag{353}$$

We want to eliminate the initial and final states from the expansions. Define a "projector" On the "rest of the state".

$$q = I - |i\rangle\langle i| - |f\rangle\langle f| \tag{354}$$

and a reduced transition operator

$$\tau(E) = V + VG_0^+(E)qV + VG_0^+(E)qVG_0^+(E)qV + \dots \tag{355}$$

τ is the transition operator from which we eliminated the singularities.

The relation between T and τ can be written starting from

$$T = V + VG_0^+T = V + VG_0^+(|i\rangle\langle i| + |f\rangle\langle f| + q)T = VS + VG_0^+qT \tag{356}$$

with

$$S = I + G_0^+(|i\rangle\langle i| + |f\rangle\langle f|)T \tag{357}$$

Renormalization of perturbation theory V

By iterating the relation $T = VS + VG_0^+ qT$ starting from $T \sim VS$

$$T = VS + VG_0^+ qVS + \ldots = \tau S \tag{358}$$

The relation between the matrix elements: we write T_{fi} and T_{ii} in terms of the matrix elements of the reduced transition operator

$$T_{fi} = \tau_{fi} + \tau_{fi} G_0^+(i) T_{ii} + \tau_{ff} G_0^+(f) T_{fi}, \tag{359}$$

$$T_{ii} = \tau_{ii} + \tau_{ii} G_0^+(i) T_{ii} + \tau_{if} G_0^+(f) T_{fi}, \tag{360}$$

In the previous equaiton we have used the notation

$$G_0^+(i/f) = (G_0^+(E))_{ii/ff} = (E - E_{i/f} + i\epsilon)^{-1} \tag{361}$$

We obtained a system of equations for T_{ii} and T_{fi} (assume $i \neq f$) from which

$$T_{fi} = \frac{\tau_{fi}}{[1 - G_0^+(i)\tau_{ii}][1 - G_0^+(f)\tau_{ff}] - \tau_{fi}\tau_{if} G_0^+(i) G_0^+(f)} \tag{362}$$

Renormalization of perturbation theory VI

The matrix element of the exact Green function G is written using $G^+(E) = G_0^+(E) + G_0^+(E)TG_0^+(E)$

$$G_{fi}^+(E) = \frac{\tau_{fi}(E)}{[E - E_i - \tau_{ii} + i\epsilon][E - E_f - \tau_{ff} + i\epsilon] - \tau_{fi}(E)\tau_{if}(E)} \tag{363}$$

Energy renormalization of the initial and final states: write the denominator in the previous equation as

$$[E - E_i - \tau_{ii} + i\epsilon][E - E_f - \tau_{ff} + i\epsilon] - \tau_{fi}(E)\tau_{if}(E) = (E - E_i' + i\epsilon)(E - E_f' + i\epsilon) \tag{364}$$

and solve for $E'_{i/f}$.

Notations:

$$E'_{i/f} = E_{i/f} + \Delta_{i/f}(E), \quad \Delta_{i/f}(E) = \tau_{ii/ff}(E) \pm \phi_{if} \tag{365}$$

$$\phi_{if}(E) = \Delta\left[\left(1 + \frac{\tau_{if}\tau_{fi}}{\Delta^2}\right)^{1/2} - 1\right], \quad \Delta = E_i + \tau_{ii} - E_f - \tau_{ff} \tag{366}$$

Renormalization of perturbation theory VII

Then

$$E'_{i/f} = E_{i/f} + \Delta_{i/f}(E) \tag{367}$$

become eigenvalue problems for E, the solutions which reduces to the unperturbed levels $E_{i/f}$ in the absence of interaction are the renormalized $E'_{i/f}$.

One obtains *complex* values for $E'_{i/f}$, "dressed energy levels" which decays.

The transition amplitude

$$\mathcal{A}_{if}(t) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE \frac{\mathcal{T}_{if}}{(E - E'_i + i\epsilon)(E - E'_f + i\epsilon)} e^{-\frac{i}{\hbar}(E-E_f)t} \tag{368}$$

Define

$$\zeta(x) = \frac{1}{x + i\epsilon} = -i\pi\delta(x) + \mathcal{P}(\frac{1}{x}) \tag{369}$$

and

$$\mathcal{A}_{if}(t) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE \mathcal{T}_{if}\zeta(E - E'_i)\zeta(E - E'_f)e^{-\frac{i}{\hbar}(E-E_f)t} \tag{370}$$

Renormalization of perturbation theory VIII

Properties of ζ function

$$\zeta(E - E_i')\zeta(E - E_f') = \zeta(E_i' - E_f')[\zeta(E - E_i') - \zeta(E - E_f')] \tag{371}$$

$$\lim_{|t|\to\infty} \zeta(E)e^{-\frac{i}{\hbar}Et} = -2\pi i\delta(E)\theta(t) \tag{372}$$

Assume the imaginary parts of E_i', E_f' are small and can be neglected. We write the transition amplitude as

$$\mathcal{A}_{fi}(\infty) = \lim_{t\to\infty} [\tau_{fi}(E_i')\zeta(E_i' - E_f')e^{-\frac{i}{\hbar}(E_i' - E_f)t} \tag{373}$$

$$-\tau_{fi}(E_f')e^{-\frac{i}{\hbar}(E_i' - E_f)t}\zeta(E_i' - E'f)e^{\frac{i}{\hbar}(E_i' - E_f')t}] \tag{374}$$

Only the first term contributes; One defines

$$C_f(t) = \mathcal{A}_{fi}(\infty)e^{\frac{i}{\hbar}(E_f' - E_f)t} \tag{375}$$

We obtain

$$C_f(t) = \tau_{if}(E_i')\zeta(E_i' - E_f')e^{-\frac{i}{\hbar}(E_i' - E_f')t} \tag{376}$$

Renormalization of perturbation theory IX

and the transition rate

$$R_{if} = \frac{d}{dt}|C(t)|^2 = \frac{2\pi}{\hbar}|\tau_{if}(E_i')|^2\delta(E_i' - E_f') \quad (377)$$