

Laser-matter interaction (FMAN)

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Chapter 1

The classical description of the electromagnetic field

1.1 The Maxwell equations

The Maxwell equations

$$\begin{aligned}\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j} &= \nabla \times \mathbf{B} \quad \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \\ \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \quad \nabla \cdot \mathbf{B} = 0.\end{aligned}\tag{1.1}$$

$\rho(\mathbf{r}, t)$ and $\mathbf{j}(\mathbf{r}, t)$ are the charge/current densities, obeying the continuity equation

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{j}(\mathbf{r}, t) = 0\tag{1.2}$$

For the electromagnetic field in vacuum the Maxwell equations reduce to the wave equation

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \Delta \mathbf{E}, \quad \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = \Delta \mathbf{B}.\tag{1.3}$$

Energy density

$$w(\mathbf{r}, t) = \frac{1}{2} \left(\epsilon_0 \mathbf{E}(\mathbf{r}, t)^2 + \frac{1}{\mu_0} \mathbf{B}(\mathbf{r}, t)^2 \right)\tag{1.4}$$

Plane wave monochromatic field (E_0 : electric field amplitude)

$$\bar{w} = \frac{1}{2} \epsilon_0 E_0^2\tag{1.5}$$

Field intensity:

$$I = \frac{1}{2} c \epsilon_0 E_0^2\tag{1.6}$$

"Atomic unit" of intensity:

$$I_a = \frac{1}{2} c \epsilon_0 E_a^2, \quad E_a = 5.1 \times 10^{11} \text{ V/m}, \quad I_a = 3.51 \times 10^{16} \text{ W/cm}^2\tag{1.7}$$

E_a is the electric field at the distance a_0 from a Hydrogen nucleus It is convenient to describe the electromagnetic field by the electromagnetic potentials

$$\mathbf{E}(\mathbf{r}, t) = -\nabla \phi(\mathbf{r}, t) - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}, \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)\tag{1.8}$$

The scalar $\Phi(\mathbf{r}, t)$ and respectively $\mathbf{A}(\mathbf{r}, t)$ vector potentials are not uniquely defined. The simultaneous change

$$\Phi(\mathbf{r}, t) \rightarrow \Phi(\mathbf{r}, t) + \frac{\partial f(\mathbf{r}, t)}{\partial t}, \quad \mathbf{A}(\mathbf{r}, t) \rightarrow \mathbf{A}(\mathbf{r}, t) - \nabla f(\mathbf{r}, t) \quad (1.9)$$

with an arbitrary continuous and derivable function of coordinates and time $f(\mathbf{r}, t)$ leaves the fields unchanged.

With these, the Maxwell equations for fields can be written as equations for potentials as

$$\nabla^2 \Phi(\mathbf{r}, t) = -\frac{\rho}{\epsilon_0}, \quad \nabla^2 \mathbf{A}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} = -\mu_0 \mathbf{j}(\mathbf{r}, t) + \nabla \cdot \left(\nabla \cdot \mathbf{A}(\mathbf{r}, t) + \frac{1}{c} \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} \right) \quad (1.10)$$

or The previous equations couple the four components of the potential A ; they have a simpler form if the potential obey the so called Lorenz condition

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) + \frac{1}{c} \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} \quad (1.11)$$

If the Lorenz condition is obeyed then the Maxwell equations reduce to four uncoupled equations for the components of the potential $\mathbf{A}(\mathbf{r}, t)$ and $\Phi(\mathbf{r}, t)$

$$\square \mathbf{A}(\mathbf{r}, t) = \mu_0 \mathbf{j}(\mathbf{r}, t), \quad \square \Phi(\mathbf{r}, t) = \frac{\rho(\mathbf{r}, t)}{\epsilon_0}, \quad \square \equiv \partial_\alpha \partial^\alpha \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \quad (1.12)$$

For the case of the electromagnetic field in vacuum the Maxwell equations become

$$\square \mathbf{A}(\mathbf{r}, t) = 0, \quad \square \Phi(\mathbf{r}, t) = 0. \quad (1.13)$$

1.2 Particular solutions of the Maxwell equations

Here we present a particular simple solution of the Maxwell equation: the plane wave. A plane wave is characterized by a propagation direction, whose unity vector will be denoted by \mathbf{n} . In most calculations the propagation direction is taken along the third axis of the reference frame ($\mathbf{n} \equiv \mathbf{e}_z$)

A plane wave, propagating along the direction \mathbf{n} can be described by a function depending on coordinates and time only through the combination

$$\phi = ct - \mathbf{n} \cdot \mathbf{r} \quad (1.14)$$

so, a plane wave solution of the Maxwell equation is defined by the scalar and vector potentials being functions on ϕ , obeying the Lorenz condition (1.11). A particular gauge used often to describe a plane-wave electromagnetic field is the so called "velocity gauge" (or "Coulomb gauge"); in this gauge the scalar potential Φ is zero and the vector potential \mathbf{A} obeys

$$\Phi(x) = 0, \quad \nabla \cdot \mathbf{A}(\phi) = 0 \quad (1.15)$$

for a plane wave the previous condition is equivalent to the requirement for \mathbf{A} to be orthogonal on the propagation direction

$$\mathbf{A}(\phi) \perp \mathbf{n} \quad (1.16)$$

The electric and magnetic fields are

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{A}(ct - \mathbf{n} \cdot \mathbf{r})}{\partial t} - \nabla \Phi(ct - \mathbf{n} \cdot \mathbf{r}) = -c \frac{d\mathbf{A}}{d\phi} + \mathbf{n} \frac{d\Phi}{d\phi} = -c \frac{d\mathbf{A}_\perp}{d\phi} \quad (1.17)$$

In the previous equation for the last equality we have used the gauge condition $\mathbf{A} \cdot \mathbf{n} = C$ and we have denoted by the index \perp the component of \mathbf{A} orthogonal on \mathbf{n} . For the magnetic field we have

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(ct - \mathbf{n} \cdot \mathbf{r}) - \mathbf{n} \times \mathbf{A}(ct - \mathbf{n} \cdot \mathbf{r}) \equiv \mathbf{n} \times \frac{d\mathbf{A}_\perp(ct - \mathbf{n} \cdot \mathbf{r})}{d\phi} \quad (1.18)$$

We see that, as expected, \mathbf{E} and \mathbf{B} are in fact functions on $\phi \equiv ct - \mathbf{n} \cdot \mathbf{r}$ and depend only on the component of \mathbf{A} orthogonal on \mathbf{n} ; this is due to the fact that the scalar potential Φ and the component of \mathbf{A} parallel to \mathbf{n} are not unique, they can be changed by a gauge transformation. The magnetic field, calculated according to (1.18) is

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \mathbf{n} \times \mathbf{E}(\mathbf{r}, t) \quad (1.19)$$

1.2.1 The monochromatic plane wave field

A particular plane wave solution of the Maxwell equation is the monochromatic plane wave; in this case the components of the vector potential and the scalar potential are periodic functions of time and coordinates. Such functions can be written as

$$A_i(x) = A_i^{(0)} \cos(|\mathbf{k}|\phi + \alpha_i^{(0)}), \quad \Phi(x) = \Phi^{(0)} \cos(|\mathbf{k}|\phi + \alpha_0^{(0)}), \quad i = 1, 2, 3 \quad (1.20)$$

where \mathbf{k} is the wave-vector

$$\mathbf{k} = \mathbf{n} \frac{2\pi}{\lambda} = \mathbf{n} \frac{\omega}{c} \quad (1.21)$$

By defining the four vector $k = (\frac{\omega}{c}, \mathbf{n}\frac{\omega}{c})$ one can write $|\mathbf{k}|\phi \equiv k \cdot x$; we also note that the Lorenz gauge condition implies that $\Phi^{(0)} = A_3^{(0)}$ and $\alpha_0^{(0)} = \alpha_3^{(0)}$. Next, we present the expressions of the electric/magnetic fields; to simplify the equations, we shall assume that the third axis of the reference frame is taken along the field propagation direction \mathbf{n} . Then we have

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{e}_x \omega A^{1,(0)} \sin(\omega t - |\mathbf{k}|z + \alpha_1^{(0)}) + \mathbf{e}_y \omega A^{2,(0)} \sin(\omega t - |\mathbf{k}|z + \alpha_2^{(0)}) \quad (1.22)$$

A typical choice of the arbitrary initial phases $\alpha_i^{(0)}$ is e.g. $\alpha_1^{(0)} = 0$, $\alpha_2^{(0)} = \pi/2$, such that

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{e}_x \omega A^{1,(0)} \sin(\omega t - |\mathbf{k}|z) + \mathbf{e}_y \omega A^{2,(0)} \cos(\omega t - |\mathbf{k}|z + \alpha_2^{(0)}) \quad (1.23)$$

If we study the vector electric field in a fixed point (e.g. the origin of the reference frame $\mathbf{E}(\mathbf{r} \equiv 0, t)$) as a function of time only, it moves along an ellipse of semi-axes $A_1^{(0)}$ and $A_2^{(0)}$; the circular polarization case corresponds to $|A_1^{(0)}| = |A_2^{(0)}|$, if one of the $A_1^{(0)}$ or $A_2^{(0)}$ is zero then we have a linearly polarized field.

1.2.2 The plane wave field; a particular case

The monochromatic solution presented before is not convenient to describe finite (especially very short duration) laser pulses. A better approximation is obtained by multiplying the monochromatic solution described above by an "envelope function", having non vanishing values only on a finite interval

$$A_i(x) = A_i^{(0)} f(|\mathbf{k}|\phi) \cos(|\mathbf{k}|\phi + \alpha_i^{(0)}), \quad \Phi(x) = \Phi^{(0)} f(|\mathbf{k}|\phi) \cos(|\mathbf{k}|\phi + \alpha_0^{(0)}), \quad i = 1, 2, 3 \quad (1.24)$$

where

$$\lim_{\rho \rightarrow \pm\infty} f(\rho) = 0; \quad (1.25)$$

a typical expression used in numerical simulations is the Gaussian profile

$$f(\rho) = e^{-[1.17\rho/(2\pi\tau)]^2} \quad (1.26)$$

It is useful to calculate the energy density

$$w(\mathbf{r}, t) = \frac{1}{2} \epsilon_0 (\mathbf{E}^2 + c^2 \mathbf{B}^2) \quad (1.27)$$

In the case described by (1.24), and if the envelope is not too short (i.e. $\tau \gg 1$) then the energy density can be approximated as

$$w(\mathbf{r}, t) = \frac{1}{2} \epsilon_0 (\mathbf{E}^2 + c^2 \mathbf{B}^2) = \frac{1}{2} \epsilon_0 \omega^2 [(A_1^{(0)})^2 + (A_2^{(0)})^2] e^{-\frac{2.35(t-z/c)^2}{\tau^2 \tau^2}} \quad (1.28)$$

written as a function of time, in the origin of the reference frame the above formula becomes also a Gaussian,

$$w(\mathbf{r} = 0, t) = \frac{1}{2} \epsilon_0 (\mathbf{E}^2 + c^2 \mathbf{B}^2) = \frac{1}{2} \epsilon_0 \omega^2 [(A_1^{(0)})^2 + (A_2^{(0)})^2] e^{-\frac{2.35t^2}{\tau^2 \tau^2}} \quad (1.29)$$

whose FWHM is $\approx 2tT$. Currently are experimentally available lasers with FWHM as short as a few optical periods (i.e. $\tau \in (1-10)$).

NB: The length of a pulse is inverse proportional with its energy bandwidth; i.e. shorter pulses are "less monochromatic"

1.2.3 The dipole approximation

In the dipole approximation the dependence on coordinates of the laser field is neglected. This approximation is valid if the process to be modeled takes place in a space region very small in comparison to the laser wave-length. A typical example is the excitation of an hydrogen atom interacting with a laser pulse in the visible/near infrared domain. In this case the laser wavelength is $\lambda \approx 10^{-6}$ m, while the typical distances involved are of the order of magnitude of the Bohr radius $a_0 \approx 5.2 \times 10^{-11}$ m.

In the dipole approximation we have

$$\mathbf{A}(\mathbf{r}, t) \rightarrow \mathbf{A}^{(d)}(t), \quad \Phi(\mathbf{r}, t) \rightarrow \Phi^{(d)}(t) \quad (1.30)$$

i.e.

$$\mathbf{E}(\mathbf{r}, t) \rightarrow \mathbf{E}(t) \equiv -\frac{d\mathbf{A}(t)}{dt}, \quad \mathbf{B}(\mathbf{r}, t) \rightarrow 0 \quad (1.31)$$

The neglecting of the coordinate dependence of the vector potential, (or, equivalently, of the radiation wave-vector $|\mathbf{k}| \equiv \omega/c \approx 0$) is in fact equivalent to taking the limit $c \rightarrow \infty$; i.e. the dipole approximation is the same as the nonrelativistic approximation.

1.2.4 Gaussian modes of the electromagnetic field

We start by solving the scalar wave equation;

$$\Delta u(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0 \quad (1.32)$$

For a monochromatic solution $u \sim e^{-i\omega t}$, $\omega = ck$ we get

$$\Delta u(\mathbf{r}, t) + k^2 = 0 \quad (1.33)$$

We look for the solutions as

$$u \sim \int_{k_x^2 + k_y^2 \leq k^2} \frac{dk_x dk_y}{(2\pi)^2} A(k_x, k_y) e^{ik_x x + ik_y y + i\sqrt{k^2 - k_x^2 - k_y^2} z} \quad (1.34)$$

We expanded the electromagnetic field in plane-waves. Note that k_z is fixed by the condition $k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$. The contribution of different values of k_x, k_y in the above expansion is fixed by the coefficients $A(k_x, k_y)$. If they are chosen such as $A(k_x, k_y) \neq 0$ only for $k_x, k_y \ll k$ (i.e. the "main" propagation direction is along Oz). Then we can use the approximation

$$\sqrt{k^2 - k_x^2 - k_y^2} \approx k - \frac{k_x^2 + k_y^2}{2k} \quad (1.35)$$

and extend the integration domain to R^2 The beam divergence is

$$\text{divergence} \sim \int_{R^2} \frac{dk_x dk_y}{(2\pi)^2} (k_x^2 + k_y^2) |A(k_x, k_y)|^2 \quad (1.36)$$

The beam transversal extension is

$$\begin{aligned} \text{trans} &\sim \int_{R^2} dx dy (x^2 + y^2) |E(x, y, z)|^2 = \\ &\int_{R^2} \frac{dk_x dk_y}{(2\pi)^2} \left(\left| \frac{dA}{dk_x} \right|^2 + \left| \frac{dA}{dk_y} \right|^2 \right) \end{aligned} \quad (1.37)$$

The "ideal beam" has minimum divergence *and* transversal extension. Then we have to minimize the products Π_x and Π_y , with

$$\Pi_x = \int_R \frac{dk_x}{(2\pi)} (k_x^2) |A(k_x, k_y)|^2 \int_R \frac{dk_x}{(2\pi)} \left(\left| \frac{dA}{dk_x} \right|^2 \right) \quad (1.38)$$

and Π_y defined similarly for the y direction. One can prove that $\Pi_x \Pi_y$ is minimized if A is a Gaussian

$$A(k_x, k_y) \sim e^{-\frac{w_0^2}{4}(k_x^2 + k_y^2)} \quad (1.39)$$

i.e.

$$u \sim \int_{\mathbb{R}^2} \frac{dk_x dk_y}{(2\pi)^2} e^{ik_x x + ik_y y + i(k - \frac{k_x^2 + k_y^2}{2k})z - \frac{w_0^2}{4}(k_x^2 + k_y^2)} \quad (1.40)$$

By direct calculation we obtain

$$u_0 \sim \frac{1}{w_0 \sqrt{1 + \zeta^2}} e^{ikz - \frac{1-i\zeta}{w_0^2(1+\zeta^2)} \rho^2 - i \arctan(\zeta)} \quad (1.41)$$

with $\zeta = z/z_R$, $z_R = kw_0^2/2$, $\rho^2 = x^2 + y^2$ i.e. a Gaussian transversal profile. This solution is named "fundamental Gaussian mode" because the corresponding light intensity

$$|u_0|^2 \sim \frac{1}{w_0^2(1 + \zeta^2)} e^{-\frac{2\rho^2}{w_0^2(1+\zeta^2)}} \quad (1.42)$$

shows a Gaussian profile perpendicularly to the propagation axis z

The Gaussian beam is the mode with minimum uncertainty, i.e. the product of its sizes in real space and wave-vector space is the theoretical minimum as given by the Heisenberg's uncertainty principle of Quantum Mechanics. Consequently, the Gaussian mode has less dispersion than any other optical field of the same size, and its diffraction sets a lower threshold for the diffraction of real optical beams. The diameter of the Gaussian beam is defined by

$$w(\zeta) = w_0^2 \sqrt{1 + \zeta^2} \quad (1.43)$$

One defines the Gouy's phase

$$\Psi_0(\zeta) = \arctan(\zeta) \quad (1.44)$$

it describes rapid phase change of the electric field when traversing the point of minimal beam diameter at $\zeta = 0$ (beam waist).

Hermite-Gaussian modes

Hermite-Gaussian beams are a family of laser modes which have rectangular symmetry along the propagation axis. They are obtained starting from

$$u \sim \int_{\mathbb{R}^2} \frac{dk_x dk_y}{(2\pi)^2} \left(ik_x + \frac{1}{v} \frac{\partial}{\partial k_x} \right)^m \left(ik_y + \frac{1}{v} \frac{\partial}{\partial k_y} \right)^n \times e^{ik_x x + ik_y y + ikz - \frac{w_0^2}{4}(1+i\zeta)(k_x^2 + k_y^2)}, \quad v = -iw_0 \quad (1.45)$$

by direct calculation one obtains

$$u_{mn}^H = \frac{1}{w(\zeta)} H_m \left(\sqrt{2} \frac{x}{w(\zeta)} \right) H_n \left(\sqrt{2} \frac{y}{w(\zeta)} \right) e^{ikz - \frac{\rho^2}{w_0^2(1+i\zeta)} - i\Psi_{n,m}} \quad (1.46)$$

with $\Psi_{n,m} = (1 + m + n) \arctan \zeta$ and H_m Hermite polynomials.

Laguerre-Gaussian modes

Differently from Hermite-Gaussian beams, Laguerre-Gaussian modes have rotational symmetry along their propagation axis and carry an intrinsic rotational orbital angular momentum

Analogous calculation, but starting from

$$u \sim \int_{\mathbb{R}^2} \frac{dk_x dk_y}{(2\pi)^2} \left(i(k_x + ik_y) + \frac{1}{v} \left(\frac{\partial}{\partial k_x} + i \frac{\partial}{\partial k_y} \right) \right)^m \left(i(k_x - ik_y) + \frac{1}{v} \left(\frac{\partial}{\partial k_x} - i \frac{\partial}{\partial k_y} \right) \right)^n \times e^{ik_x x + ik_y y + ikz - \frac{w_0^2}{4}(1+i\zeta)(k_x^2 + k_y^2)}, \quad v = -iw_0 \quad (1.47)$$

with the result

$$u_{mn}^L = \frac{e^{-im\phi}}{w(\zeta)} \left(\frac{\rho}{w(\zeta)} \right)^n L_m^n \left(\frac{2\rho^2}{w^2(\zeta)} \right) e^{ikz - \frac{\rho^2}{w_0^2(1+i\zeta)} - i\psi_{n,m}^L} \quad (1.48)$$

Here $\psi_{n,m} = (1 + 2m + n) \arctan \zeta$ and L_m^n are Laguerre polynomials. The constant phase surface have an helical form, i.e. the field carries angular momentum.

The electric and magnetic fields of a Gaussian mode

Using the previous solutions Hermite-Gauss (1.46) of Laguerre-Gauss (1.48) of the wave equation, denoted here in general by $u(\mathbf{r})$, we can build the electric and magnetic fields. In order to do so, we start from the electric field, whose x and y components can be independent solutions $u(\mathbf{r}, t)$ multiplied by $e^{-i\omega_0 t}$. The third component E_z must then taken such that $\nabla \cdot \mathbf{E} = 0$. With these we get

$$\mathbf{E}(\mathbf{r}, t) = e^{-i\omega_0 t} \left[\alpha u(\mathbf{r}) \mathbf{e}_x + \beta u(\mathbf{r}) \mathbf{e}_y + \frac{i}{k} \left(\alpha \frac{\partial u(\mathbf{r})}{\partial x} + \beta \frac{\partial u(\mathbf{r})}{\partial y} \right) \mathbf{e}_z \right]. \quad (1.49)$$

Next, the magnetic field is completely determined by the condition

$$-c \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \quad (1.50)$$

which, considering the dependence on t as $e^{-i\omega t}$ gives

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{i\omega_0} \nabla \times \mathbf{E}(\mathbf{r}, t). \quad (1.51)$$

Finally, we mention that in calculations must be used the real part of the complex solutions defined before.

Chapter 2

The classical trajectory of a charged particle in an electromagnetic plane wave field

2.1 NR dipole approximation

We shall use the velocity gauge:

$$\mathbf{A}(t) = f(t) [\zeta_x \mathbf{e}_x \sin(\omega t) + \zeta_y \mathbf{e}_y \cos(\omega t)], \quad \zeta_x^2 + \zeta_y^2 = 1 \quad (2.1)$$

with $f(t)$ arbitrary, and $\lim_{t \pm \infty} f(t) = 0$. The electric field is

$$\mathbf{E}(t) = -\frac{d\mathbf{A}(t)}{dt}, \quad (2.2)$$

and the magnetic field is neglected. The equation of motion leads to

$$m\dot{\mathbf{r}}(t) = e\mathbf{E}(t), \quad m\mathbf{r}(t_0) = 0, \quad m\dot{\mathbf{r}}(t_0) = \mathbf{v}_0 \quad (2.3)$$

whose solution is

$$m\dot{\mathbf{r}}(t) = -e\mathbf{A}(t) + \mathbf{v}_0, \quad \mathbf{r}(t) = -\frac{e}{m} \int_{t_0}^t \mathbf{A}(t') dt' + \mathbf{v}_0(t - t_0) \quad (2.4)$$

With the notation

$$-\frac{e}{m} \int_{t_0}^t \mathbf{A}(t') dt' = \boldsymbol{\alpha}(t) \quad (2.5)$$

we have

$$\mathbf{r}(t) = \boldsymbol{\alpha}(t) + \mathbf{v}_0(t - t_0) \quad (2.6)$$

2.2 NR non-dipole approximation

We shall consider *only* plane-wave fields defined in (1.20), in the velocity gauge $\Phi = 0$, $\nabla \times \mathbf{A} = 0$, and we shall assume that at a given moment t_0 the particle is in the origin of the reference frame, and it has the given velocity \mathbf{v}_0 . We denote by \mathbf{A}_0 the value of the potential in the origin of the reference frame at the initial moment t_0 , with $\phi_0 = ct_0$. For the case of a pulse, the most natural choice for t_0 is a moment sufficiently far in the past, when the pulse has not reached the origin and the particle is free (i.e. $A_0 = 0$).

Cases of interest:

Case 1: linearly polarized pulse; the particle is at rest in the origin at a moment t_0 very far in the past, when the pulse has not yet reached the origin.

Case 2: linearly polarized monochromatic field; at the moment t_0 when the field vanishes in the origin, the particle is located in the origin, with an initial velocity \mathbf{v}_0 directed along the field propagation direction.

The Lagrange function of a nonrelativistic particle of mass m and electric charge e moving in the electromagnetic field $A(\phi)$

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{m\dot{\mathbf{r}}^2}{2} + e\dot{\mathbf{r}} \cdot \mathbf{A}(\phi) \quad (2.7)$$

gives the equations of motion

$$\frac{d}{dt} (m\dot{\mathbf{r}}_{\perp} + e\mathbf{A}(\phi)) = 0, \quad (2.8)$$

$$\frac{d}{dt} (m\dot{z}) = e\dot{\mathbf{r}} \cdot \frac{\partial \mathbf{A}(\phi)}{\partial z} = -e\dot{\mathbf{r}} \cdot \frac{d\mathbf{A}(\phi)}{d\phi}, \quad (2.9)$$

where \mathbf{r}_{\perp} is the component of \mathbf{r} orthogonal on the propagation direction $\mathbf{n} \equiv \mathbf{e}_z$. From the first of the two equations above, and taking into account the initial conditions one obtains

$$\dot{\mathbf{r}}_{\perp} = \mathbf{v}_{0\perp} - \frac{e}{m}(\mathbf{A}(\phi) - \mathbf{A}_0). \quad (2.10)$$

Using this result in Eq.(2.9) one gets

$$\frac{d}{dt} \dot{z} = \frac{d}{d\phi} \left[\frac{e^2}{2m^2} \left(\mathbf{A}(\phi) - \mathbf{A}_0 - \frac{m}{e} \mathbf{v}_{0\perp} \right)^2 \right] \quad (2.11)$$

It is convenient to use ϕ as the independent variable. From the relation

$$\frac{d\phi}{dt} = c \left(1 - \frac{\dot{z}}{c} \right) \quad (2.12)$$

one can see that this change of variable only makes sense if $\dot{z} < c$; the restriction is however not a problem, since we are not interested, anyway, to find a solution with $\dot{z} \geq c$. The above equation in the new variable ϕ writes as

$$\frac{d}{d\phi} \left(\frac{\dot{z}}{c} - \frac{1}{2} \frac{\dot{z}^2}{c^2} \right) = \frac{d}{d\phi} \left[\frac{e^2}{2(mc)^2} \left(\mathbf{A}(\phi) - \mathbf{A}_0 - \frac{m}{e} \mathbf{v}_{0\perp} \right)^2 \right]. \quad (2.13)$$

Taking into account the initial conditions one obtains the equation

$$\frac{\dot{z}}{c} - \frac{1}{2} \frac{\dot{z}^2}{c^2} = \frac{e^2 \mathcal{A}^2(\phi)}{2(mc)^2} + \frac{v_{03}}{c} - \frac{1}{2} \frac{v_{03}^2}{c^2}, \quad (2.14)$$

with

$$\mathcal{A}^2(\phi) = (\mathbf{A}(\phi) - \mathbf{A}_0)^2 - \frac{2m\mathbf{v}_{0\perp}}{e} \cdot (\mathbf{A}(\phi) - \mathbf{A}_0) \quad (2.15)$$

whose solution is

$$\frac{\dot{z}}{c} = 1 - \sqrt{\left(1 - \frac{v_{03}}{c}\right)^2 - \frac{e^2 \mathcal{A}^2(\phi)}{(mc)^2}}. \quad (2.16)$$

One notices that the solution \dot{z} is defined only if

$$\frac{e^2 \mathcal{A}^2(\phi)}{2(mc)^2} < \left(1 - \frac{v_{03}}{c}\right)^2 \quad (2.17)$$

and that \dot{z} cannot become larger than c . Again, using ϕ as the independent variable, one gets

$$\frac{dz}{d\phi} = \frac{1}{c - \frac{dz}{dt} \frac{d\phi}{dt}}, \quad \frac{d\mathbf{r}_{\perp}}{d\phi} = \frac{1}{c - \frac{dz}{dt} \frac{d\phi}{dt}} \frac{d\mathbf{r}_{\perp}}{dt} \quad (2.18)$$

or

$$z = \int_{\phi_0}^{ct-z} d\chi \frac{1 - \sqrt{\left(1 - \frac{v_{03}}{c}\right)^2 - \frac{e^2 \mathcal{A}^2(\chi)}{(mc)^2}}}{\sqrt{\left(1 - \frac{v_{03}}{c}\right)^2 - \frac{e^2 \mathcal{A}^2(\chi)}{(mc)^2}}}, \quad \mathbf{r}_{\perp} = -\frac{e}{mc} \int_{\phi_0}^{ct-z} d\chi \frac{\mathbf{A}(\chi) - \mathbf{A}_0 - \frac{m\mathbf{v}_{\perp}}{e}}{\sqrt{\left(1 - \frac{v_{03}}{c}\right)^2 - \frac{e^2 \mathcal{A}^2(\chi)}{(mc)^2}}}, \quad \phi_0 = ct_0. \quad (2.19)$$

The first of the above equations must be solved for z , then its solution used in the second one, to get r_{\perp} .

Examples:

Case 1 is described by the conditions: $\phi_0 = -\infty$, $\mathbf{A}_0 = 0$, $\mathbf{v}_0 = 0$, the corresponding trajectory being:

$$z = \int_{-\infty}^{ct-z} d\chi \frac{1 - \sqrt{1 - \frac{e^2 \mathbf{A}^2(\chi)}{(mc)^2}}}{\sqrt{1 - \frac{e^2 \mathbf{A}^2(\chi)}{(mc)^2}}}, \quad r_{\perp} = -\frac{e}{mc} \int_{\phi_0}^{ct-z} d\chi \frac{\mathbf{A}(\chi)}{\sqrt{1 - \frac{e^2 \mathbf{A}^2(\chi)}{(mc)^2}}}. \quad (2.20)$$

The solution is defined for $e^2 \mathbf{A}^2(\phi) < (mc)^2$; in fact, the non-relativistic approximation is valid only in the limit $e^2 \mathbf{A}^2(\phi) \ll (mc)^2$. In this case the previous solution becomes

$$z = \int_{-\infty}^{ct-z} d\chi \frac{e^2 \mathbf{A}^2(\chi)}{2(mc)^2}, \quad r_{\perp} = -\frac{e}{mc} \int_{\phi_0}^{ct-z} d\chi \mathbf{A}(\chi). \quad (2.21)$$

In the **case 2** we have $\phi_0 = 0$, $\mathbf{A}_0 = 0$, $\mathbf{v}_0 = v_{03} \mathbf{n}$, and the corresponding solution

$$z = \int_0^{ct-z} d\chi \frac{1 - \sqrt{\left(1 - \frac{v_{03}}{c}\right)^2 - \frac{e^2 \mathbf{A}^2(\chi)}{(mc)^2}}}{\sqrt{\left(1 - \frac{v_{03}}{c}\right)^2 - \frac{e^2 \mathbf{A}^2(\chi)}{(mc)^2}}}, \quad r_{\perp} = -\frac{e}{mc} \int_0^{ct-z} d\chi \frac{\mathbf{A}(\chi)}{\sqrt{\left(1 - \frac{v_{03}}{c}\right)^2 - \frac{e^2 \mathbf{A}^2(\chi)}{(mc)^2}}}. \quad (2.22)$$

In this case the conditions of validity of the non-relativistic approximation are $e^2 \mathbf{A}^2(\phi) \ll (mc)^2$ and $v_{03} \ll c$, which leads to the trajectory

$$z = \int_{-\infty}^{ct-z} d\chi \left[\frac{v_{03}}{c} + \frac{e^2 \mathbf{A}^2(\chi)}{2(mc)^2} \right], \quad r_{\perp} = -\frac{e}{mc} \int_{\phi_0}^{ct-z} d\chi \mathbf{A}(\chi). \quad (2.23)$$

Chapter 3

The quantization of the electromagnetic field

3.1 Field quantisation in the Coulomb gauge

We associate to the quantized electromagnetic field a Hilbert space and to the physical hermitian operators in the Hilbert space. Start from the Coulomb gauge (1.15). We consider the electromagnetic field in a box of volume $V = L^3$, and impose periodicity conditions on the fields. In the *Schrodinger picture* we have for a single *mode* of the field

$$\mathbf{A}_{\text{mode}}^{\text{op}}(\mathbf{r}) = \sqrt{\frac{\hbar}{2\epsilon_0\omega V}} [\mathbf{s} e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} \hat{a} + \mathbf{s}^* e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}} \hat{a}^\dagger], \quad (3.1)$$

where the mode is characterised by the wavevector $\boldsymbol{\kappa}$, $\omega = c|\boldsymbol{\kappa}|$, and the polarization vector \mathbf{s} , with $\mathbf{s} \cdot \boldsymbol{\kappa} = 0$, and \hat{a}^\dagger , \hat{a} are the creation/annihilation operators obeying

$$[\hat{a}, \hat{a}^\dagger] = \hat{1}. \quad (3.2)$$

The electric field operator will be

$$\mathbf{E}_{\text{mode}}^{\text{op}}(\mathbf{r}) = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} i [\mathbf{s} e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} \hat{a} - \mathbf{s}^* e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}} \hat{a}^\dagger], \quad (3.3)$$

and the magnetic field operator

$$\mathbf{B}_{\text{mode}}^{\text{op}}(\mathbf{r}) = i \sqrt{\frac{\hbar}{2\epsilon_0 V \omega}} \boldsymbol{\kappa} \times [\mathbf{s} e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} \hat{a} - \mathbf{s}^* e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}} \hat{a}^\dagger]. \quad (3.4)$$

Operators for the entire field are sums over all modes

$$\mathbf{A}^{\text{op}}(\mathbf{r}) = \sum_{\text{mode}} \mathbf{A}_{\text{mode}}^{\text{op}}(\mathbf{r}). \quad (3.5)$$

In the previous sum we have all the possible values of the wavevector $\boldsymbol{\kappa}$ allowed by the periodicity conditions, and for each $\boldsymbol{\kappa}$ two polarization vectors \mathbf{s}_1 , \mathbf{s}_2 , with $\mathbf{s}_1^* \cdot \mathbf{s}_2 = 0$. Different oscillation modes are independent, i.e. the corresponding operators commute

$$[\hat{a}_{\text{mode}_1}, \hat{a}_{\text{mode}_2}] = [\hat{a}_{\text{mode}_1}, \hat{a}_{\text{mode}_2}^\dagger] = 0. \quad (3.6)$$

The corresponding commutation relations for the field operators are

$$\begin{aligned} [(E_{\text{mode}}^{\text{op}})_j(\mathbf{r}_1), (E_{\text{mode}}^{\text{op}})_k(\mathbf{r}_2)] &= -\frac{i\hbar\omega}{\epsilon_0 V} s_j s_k \sin(\boldsymbol{\kappa} \cdot (\mathbf{r}_1 - \mathbf{r}_2)) \hat{1}, \\ [(E_{\text{mode}}^{\text{op}})_j(\mathbf{r}_1), (B_{\text{mode}}^{\text{op}})_k(\mathbf{r}_2)] &= \frac{i\hbar}{\epsilon_0 V} s_j (\boldsymbol{\kappa} \times \mathbf{s})_k \cos(\boldsymbol{\kappa} \cdot (\mathbf{r}_1 - \mathbf{r}_2)) \hat{1}, \end{aligned} \quad (3.7)$$

where i, j are the indices of the Cartesian components, $i, j \in \{x, y, z\}$.

3.1.1 The energy operator; the Fock space

The energy operator of the field in the volume V is, up to an (infinite) constant

$$\hat{H}_{\text{rad}} = \sum_{\text{mode}} \hat{H}_{\text{mode}}^{\text{op}} = \sum_{\text{mode}} \hbar\omega_{\text{mode}} \hat{a}_{\text{mode}}^{\dagger} \hat{a}_{\text{mode}} . \quad (3.8)$$

The contribution of each mode to the total energy operator is

$$H_{\text{mode}}^{\text{op}} = \hat{a}^{\dagger} \hat{a} \hbar\omega . \quad (3.9)$$

whose eigenvalue problem is

$$\hat{H}_{\text{mode}}^{\text{op}} |n; \text{mode}\rangle = n\hbar\omega |n; \text{mode}\rangle , \quad n \text{ integer } \geq 0 . \quad (3.10)$$

In the previous equation there are the energy eigenvectors $|n; \text{mode}\rangle$. The value of the number n is named the occupation number, or the photon number in the mode. The energy eigenvectors are orthogonal and normalized,

$$\langle n; \text{mode} | n'; \text{mode}\rangle = \delta_{nn'} . \quad (3.11)$$

Sometimes, for simplicity we shall use the short notation $|n; \text{mode}\rangle \rightarrow |n\rangle$. One defines the operator of the number of particles in a given mode

$$\hat{N}_{\text{mode}} \equiv \hat{a}_{\text{mode}}^{\dagger} \hat{a}_{\text{mode}} . \quad (3.12)$$

whose eigenvectors are the same as the energy eigenvectors

$$|n\rangle = \frac{(\hat{a}^{\dagger})^n}{\sqrt{n!}} |0\rangle . \quad (3.13)$$

Assuming that $|0\rangle$ is normalized, then all the vectors defined in the previous relation will be also normalized. It is useful to mention the action of the annihilation/creation operators

$$\begin{aligned} \hat{a} |n\rangle &= \sqrt{n} |n-1\rangle , \quad (n \neq 0), \quad \hat{a} |0\rangle = | \text{zero} \rangle , \\ \hat{a}^{\dagger} |n\rangle &= \sqrt{n+1} |n+1\rangle . \end{aligned} \quad (3.14)$$

where $| \text{zero} \rangle$ is the null vector in the Fock space.

Considering now all the oscillation modes, the total states space is the direct product of the spaces for each mode, a basis being

$$| \text{Fock} \rangle \equiv \prod_{\text{mode}} |n; \text{mode}\rangle . \quad (3.15)$$

A vector from of the type written above is characterized by a set of numbers, representing the occupation number in each oscillation mode; an equivalent notation for such a vector is

$$| \text{Fock} \rangle = |n_1, n_2, \dots, n_p, \dots\rangle . \quad (3.16)$$

Such a vector is also an energy eigenvector, for the eigenvalue

$$W_{\text{Fock}} = \sum_{\text{mode}} n_{\text{mode}} \hbar\omega_{\text{mode}} . \quad (3.17)$$

Also the Fock vectors are normalized and orthogonal.

The stationary states of the electromagnetic field are

$$| \psi_{\text{stat}}^{\text{Fock}} \rangle = | \text{Fock} \rangle e^{-\frac{i}{\hbar} W_{\text{Fock}} t} . \quad (3.18)$$

The *vacuum state* is the state with all the occupation numbers zero. The energy density in a Fock state is

$$\rho_V = \frac{1}{V} \sum_{\text{mode}} n_{\text{mode}} \hbar\omega_{\text{mode}} .$$

If we take the limit $V \rightarrow \infty$, using

$$\frac{1}{V} \sum_{\text{mode}} \dots \rightarrow \frac{1}{(2\pi)^3} \int_{\kappa} \sum_{\lambda=1,2} \dots d\kappa .$$

we get

$$\rho = \frac{1}{(2\pi)^3} \int d\Omega_{\kappa} \sum_{\lambda=1,2} \int_0^{\infty} \kappa^2 n(\kappa, \mathbf{s}_{\lambda}) \hbar \omega d\kappa ,$$

where n_{mode} was replaced by $n(\kappa, \mathbf{s}_{\lambda})$, with κ real. The index λ indicates the polarization state of the mode. Then

$$\rho(\kappa, \mathbf{s}_{\lambda}) = \frac{\hbar \omega^3}{(2\pi c)^3} n(\kappa, \mathbf{s}_{\lambda}) . \quad (3.19)$$

is the spectral energy density with fixed polarization.

One can calculate the expectation value of the electric and magnetic field in a Fock state

$$\begin{aligned} \bar{\mathbf{E}}(\mathbf{r}) &\equiv \langle \psi_{\text{stat}}^{\text{Fock}} | \mathbf{E}^{\text{op}} | \psi_{\text{stat}}^{\text{Fock}} \rangle = \langle \text{Fock} | \mathbf{E}^{\text{op}} | \text{Fock} \rangle = 0 , \\ \bar{\mathbf{B}}(\mathbf{r}) &\equiv \langle \psi_{\text{stat}}^{\text{Fock}} | \mathbf{B}^{\text{op}} | \psi_{\text{stat}}^{\text{Fock}} \rangle = 0 . \end{aligned} \quad (3.20)$$

the result shows that the pure Fock states would lead to vanishing observed values for the fields, i.e. pure fock states are not observed in usual conditions. The standard deviation of the electric field is define as

$$\Delta E \equiv \sqrt{\langle \psi | \mathbf{E}^{\text{op}} \cdot \mathbf{E}^{\text{op}} | \psi \rangle - \langle \psi | \mathbf{E}^{\text{op}} | \psi \rangle \cdot \langle \psi | \mathbf{E}^{\text{op}} | \psi \rangle} . \quad (3.21)$$

and for the monomode case the result is

$$\Delta E_{\text{mode}} = \sqrt{\frac{\hbar \omega}{\epsilon_0 V}} (n + 1/2) . \quad (3.22)$$

The Heisenberg picture

In the *Heisenberg picture*, obtained from the Schrödinger picture using the transformation operator

$$\hat{U}_{S \rightarrow H}(t) = e^{\frac{i}{\hbar} \hat{H}_{\text{rad}} t} .$$

the state vectors are time independet

$$| \psi_{\text{stat}, \text{Fock}}^H \rangle = | \text{Fock} \rangle , \quad (3.23)$$

and the field operators are

$$\hat{a}^H(t) = \hat{U}_{S \rightarrow H} \hat{a} \hat{U}_{S \rightarrow H}^\dagger = \hat{a} e^{-i\omega t}, \quad (\hat{a}^H(t))^\dagger = \hat{a}^\dagger e^{i\omega t} \quad (3.24)$$

The vector potential then has the form

$$\mathbf{A}^{\text{op},H}(\mathbf{r}, t) = \sum_{\text{mode}} \sqrt{\frac{\hbar}{2\epsilon_0 V \omega}} [\mathbf{s} e^{i(\kappa \cdot \mathbf{r} - \omega t)} \hat{a}_{\text{mode}} + \mathbf{s}^* e^{-i(\kappa \cdot \mathbf{r} - \omega t)} \hat{a}_{\text{mode}}^\dagger] . \quad (3.25)$$

3.2 Coherent states

Definition and basic properties

We consider only on mode of the field, denoting the creation and annihilation operators by a^\dagger and a . A Coherent state is an eigenstate of the operator a

$$\hat{a} | z \rangle = z | z \rangle . \quad (3.26)$$

To solve the previous equation we look for $|z\rangle$ as

$$|z\rangle = \sum_{n=0}^{\infty} c_n |n\rangle . \quad (3.27)$$

and we get the recursion relation

$$\sqrt{n+1} c_{n+1} = z c_n , \quad (3.28)$$

from which we can write

$$c_n = \frac{z^n}{\sqrt{n!}} c_0 . \quad (3.29)$$

The normalization condition

$$\langle z | z \rangle = 1 , \quad (3.30)$$

leads to

$$|c_0|^2 \sum_{n=0}^{\infty} \frac{|z|^2}{n!} = |c_0|^2 e^{|z|^2} = 1 , \quad (3.31)$$

and we are led to $|c_0|^2 = e^{-|z|^2}$, si we can choose

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle , \quad (3.32)$$

which shall be use in the following.

We notice here that the creation operator has no eigenstates.

The scalar product between two coherent states si

$$\langle z_1 | z_2 \rangle = \exp \left[-\frac{1}{2} |z_1 - z_2|^2 + i \operatorname{Im}(z_1^* z_2) \right] . \quad (3.33)$$

i.e. two states corresponding to different eigenvalues are not orthogonal (the property would be valid if a was hermitian, which is not the case).

For a monomode electromagnetic field in a coherent state,

$$|\Psi(t_0)\rangle = |z\rangle \quad (3.34)$$

the number of photons in the given mode $|z\rangle$ is not determined. The probability to have n photons is

$$p(n) = |c_n|^2 = e^{-|z|^2} \frac{|z|^{2n}}{n!} . \quad (3.35)$$

The above distribution law is the Poisson law, and the average number of photons becomes

$$\bar{n} = |z|^2 . \quad (3.36)$$

This, together with the eh expectation value of the square of the photon number

$$\overline{n^2} = |z|^4 + |z|^2 ,$$

leads to the standard deviation

$$\overline{(\delta n)^2} = \bar{n} = |z|^2 . \quad (3.37)$$

An important property of a coherent state is: *A coherent state at the moment t_0 , with the eigenvalue z of the operator \hat{a} , evolves in a coherent state with the eigenvalue $z \exp[-i\omega(t - t_0)]$.* The proof of the previous property is based on the identity

$$|\Psi(t_0)\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \longrightarrow |\Psi(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-\frac{i}{\hbar} E_n(t-t_0)} |n\rangle .$$

applied for the state (3.32) with $E_n = n \hbar \omega$,

$$|\psi(t)\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{(ze^{-i\omega(t-t_0)})^n}{\sqrt{n!}} |n\rangle \equiv |ze^{-i\omega(t-t_0)}\rangle. \quad (3.38)$$

We shall use the notation

$$z(t) \equiv ze^{-i\omega(t-t_0)}. \quad (3.39)$$

The electromagnetic field in a coherent state

In a coherent state the expectation value of the electric field is

$$\overline{\mathbf{E}(t)} \equiv \langle z(t) | \mathbf{E}^{\text{op}}(\mathbf{r}) | z(t) \rangle = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} i [ze^{i(\kappa \cdot \mathbf{r} - \omega(t-t_0))} \mathbf{s} - z^* e^{-i(\kappa \cdot \mathbf{r} - \omega(t-t_0))} \mathbf{s}^*]. \quad (3.40)$$

We can see the same dependence on time and coordinates as in the classical solution (1.22); a similar expression is found for the magnetic field.

The coherent states are used for describing particular states of the electromagnetic field which are the closest to the classical states.

Coherent states as minimal states

One can define the combinations of the creation/annihilation operators

$$\hat{Q} \equiv \left(\frac{\hbar}{2\omega}\right)^{1/2} (\hat{a}^\dagger + \hat{a}), \quad \hat{P} \equiv i \left(\frac{\hbar\omega}{2}\right)^{1/2} (\hat{a}^\dagger - \hat{a}). \quad (3.41)$$

whose commutation relations are

$$[\hat{Q}, \hat{P}] = i\hbar, \quad (3.42)$$

the same as for the position/momentum operators.

One can easily see that in a coherent state we get the results

$$\begin{aligned} \langle z | \hat{Q} | z \rangle &= \frac{2\hbar}{\omega} \text{Re}(z), & \langle z | \hat{P} | z \rangle &= 2\hbar\omega \text{Im}(z), \\ \overline{(\delta Q)^2} &= \hbar/2\omega, & \overline{(\delta P)^2} &= \hbar\omega/2, \end{aligned} \quad (3.43)$$

from which we have $\Delta Q \cdot \Delta P = \hbar/2$, i.e. the minimum value allowed by the Heisenberg condition, i.e. the coherent states are minimal states.

Chapter 4

Atomic systems in electromagnetic fields

4.1 The continuity equation

Consider an one electron atom in a classical electromagnetic field described by the potentials $\phi(\mathbf{r}, t)$, $\mathbf{A}(\mathbf{r}, t)$. The Hamilton system is

$$H = \frac{(\mathbf{P} - e\mathbf{A}(\mathbf{r}, t))^2}{2m_e} + V(\mathbf{r}) + e\phi(\mathbf{r}, t) \quad (4.1)$$

In the previous equation, we have denoted by m_e and e the mass and respectively the charge of the electron and with $V(\mathbf{r})$ the atomic potential. Equivalent forms of the Hamiltonian are

$$H = \frac{\mathbf{P}^2 + e^2\mathbf{A}^2(\mathbf{r}, t) - e\mathbf{A}(\mathbf{r}, t) \cdot \mathbf{P} - \mathbf{P} \cdot e\mathbf{A}(\mathbf{r}, t)}{2m_e} + V(\mathbf{r}) + e\phi(\mathbf{r}, t), \quad (4.2)$$

$$H = \frac{\mathbf{P}^2 + e^2\mathbf{A}^2(\mathbf{r}, t) - 2e\mathbf{A}(\mathbf{r}, t) \cdot \mathbf{P} + (i\hbar\nabla \cdot e\mathbf{A}(\mathbf{r}, t))}{2m_e} + V(\mathbf{r}) + e\phi(\mathbf{r}, t). \quad (4.3)$$

In order to obtain the continuity equation we write the Schrödinger equation and its conjugate

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = \frac{\mathbf{P}^2 + e^2\mathbf{A}^2(\mathbf{r}, t) - 2e\mathbf{A}(\mathbf{r}, t) \cdot \mathbf{P} + (i\hbar\nabla \cdot e\mathbf{A}(\mathbf{r}, t))}{2m_e} \Psi(\mathbf{r}, t) + V(\mathbf{r})\Psi(\mathbf{r}, t) + e\phi(\mathbf{r}, t)\Psi(\mathbf{r}, t), \quad (4.4)$$

$$-i\hbar \frac{\partial \Psi^*(\mathbf{r}, t)}{\partial t} = \frac{\mathbf{P}^2 + e^2\mathbf{A}^2(\mathbf{r}, t) - 2e\mathbf{A}(\mathbf{r}, t) \cdot \mathbf{P} - (i\hbar\nabla \cdot e\mathbf{A}(\mathbf{r}, t))}{2m_e} \Psi^*(\mathbf{r}, t) + V(\mathbf{r})\Psi^*(\mathbf{r}, t) + e\phi(\mathbf{r}, t)\Psi^*(\mathbf{r}, t). \quad (4.5)$$

and by combining them we get

$$i\hbar \frac{\partial |\Psi(\mathbf{r}, t)|^2}{\partial t} = -\frac{\hbar^2}{2m_e} [\Psi^*(\mathbf{r}, t)\Delta\Psi(\mathbf{r}, t) - \Psi(\mathbf{r}, t)\Delta\Psi^*(\mathbf{r}, t)] + \frac{i\hbar e}{m_e} \nabla \cdot \mathbf{A}(\mathbf{r}, t) |\Psi(\mathbf{r}, t)|^2 \quad (4.6)$$

or

$$i\hbar \frac{\partial |\Psi(\mathbf{r}, t)|^2}{\partial t} = -i\hbar \nabla \cdot \left[\frac{\hbar}{2im_e} [\Psi^*(\mathbf{r}, t)\nabla\Psi(\mathbf{r}, t) - \Psi(\mathbf{r}, t)\nabla\Psi^*(\mathbf{r}, t)] - \frac{e\mathbf{A}(\mathbf{r}, t)}{m_e} |\Psi(\mathbf{r}, t)|^2 \right] \quad (4.7)$$

The previous result can be read as the continuity equation

$$\frac{\partial \mathcal{P}(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) = 0 \quad (4.8)$$

with the probability density

$$\mathcal{P}(\mathbf{r}, t) = |\Psi(\mathbf{r}, t)|^2 \quad (4.9)$$

and the density current

$$\mathbf{J}(\mathbf{r}, t) = \frac{\hbar}{2im_e} [\Psi^*(\mathbf{r}, t)\nabla\Psi(\mathbf{r}, t) - \Psi(\mathbf{r}, t)\nabla\Psi^*(\mathbf{r}, t)] - \frac{e\mathbf{A}(\mathbf{r}, t)}{m_e} |\Psi(\mathbf{r}, t)|^2 \quad (4.10)$$

4.2 The non-relativistic Volkov solutions

Consider the electromagnetic field in the dipole approximation described in the velocity gauge by the vector potential $\mathbf{A}(t)$ and a particle of mass m_e and charge e ; the Schrödinger equation is

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = \frac{(\mathbf{P} - e^2 \mathbf{A}^2(t))^2}{2m_e} \Psi(\mathbf{r}, t) \quad (4.11)$$

We look for the solution as the plane wave multiplied by a function of time

$$\Psi(\mathbf{r}, t) = \frac{1}{(2\pi\hbar)^{3/2}} e^{i\mathbf{p}\cdot\mathbf{r}} f(t) \quad (4.12)$$

The equation for $f(t)$ is

$$i\hbar \frac{df(t)}{dt} = \frac{(\mathbf{p} - e^2 \mathbf{A}^2(t))^2}{2m_e} f(t) \quad (4.13)$$

with the solution

$$f(t) = e^{-\frac{i}{\hbar} \int dt' \frac{(\mathbf{p} - e^2 \mathbf{A}^2(t'))^2}{2m_e}} \quad (4.14)$$

The complete solutions

$$\Psi(\mathbf{p}; \mathbf{r}, t) = \frac{1}{(2\pi\hbar)^{3/2}} e^{i\mathbf{p}\cdot\mathbf{r}} e^{-\frac{i}{\hbar} \int dt' \frac{(\mathbf{p} - e^2 \mathbf{A}^2(t'))^2}{2m_e}} \quad (4.15)$$

are named Volkov solutions; they are eigenvectors of the momentum operator, for the eigenvalue \mathbf{p} . In the limit $\mathbf{A}(t) \rightarrow 0$ the Volkov solutions reduce to the known plane wave solutions

$$\Psi_0(\mathbf{p}; \mathbf{r}, t) = \frac{1}{(2\pi\hbar)^{3/2}} e^{i\mathbf{p}\cdot\mathbf{r}} e^{-\frac{i}{\hbar} \frac{\mathbf{p}^2}{2m_e} t} \quad (4.16)$$

The Volkov solutions obey also the orthogonality

$$\langle \Psi(\mathbf{p}, \mathbf{r}, t) | \Psi(\mathbf{p}', \mathbf{r}, t) \rangle = \delta(\mathbf{p} - \mathbf{p}') \quad (4.17)$$

and completeness relations

$$\int d\mathbf{p} |\Psi(\mathbf{p}, \mathbf{r}, t)\rangle \langle \Psi(\mathbf{p}, \mathbf{r}', t)| = \delta(\mathbf{r} - \mathbf{r}') \quad (4.18)$$

at the same moment, i.e. the for a basis set in the Hilbert space.

4.3 The Schrödinger equation for the atom in the electromagnetic field in the dipole approximation

In the dipole approximation, and in the velocity gauge the field is described by the time-dependent vector potential $\mathbf{A}(t)$ and the Schrödinger equation (4.4) reduces to

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = \frac{\mathbf{P}^2 + e^2 \mathbf{A}^2(t) - 2e\mathbf{A}(t) \cdot \mathbf{P}}{2m_e} \Psi(\mathbf{r}, t) + V(\mathbf{r}) \Psi(\mathbf{r}, t), \quad (4.19)$$

The term containing the square of the vector potential can be easily eliminated by an unitary transformation

$$\Psi(\mathbf{r}, t) \rightarrow \tilde{\Psi}(\mathbf{r}, t) = \Psi(\mathbf{r}, t) e^{-\frac{i}{\hbar} \frac{e^2 \mathbf{A}^2(t)}{2m_e}} \quad (4.20)$$

the equation for the new function being

$$i\hbar \frac{\partial \tilde{\Psi}(\mathbf{r}, t)}{\partial t} = \frac{\mathbf{P}^2 - 2e\mathbf{A}(t) \cdot \mathbf{P}}{2m_e} \tilde{\Psi}(\mathbf{r}, t) + V(\mathbf{r}) \tilde{\Psi}(\mathbf{r}, t), \quad (4.21)$$

The Schrodinger equation can be written also in the length gauge, in which the field is described only by the scalar potential

$$\Phi(\mathbf{r}, t) = \frac{d\mathbf{A}(t)}{dt} \cdot \mathbf{r}; \quad (4.22)$$

as

$$i\hbar \frac{\partial \psi_i(\mathbf{r}, t)}{\partial t} = \frac{\mathbf{P}^2}{2m} \psi_i(\mathbf{r}, t) + [e\Phi(\mathbf{r}, t) + V(\mathbf{r})] \psi_i(\mathbf{r}, t), \quad (4.23)$$

The last form of the Schrödinger equation presented here is the so called *Kramers-Henneberger form*; it is obtained from the velocity gauge equation in the form (4.21) performing the transformation

$$\Psi_{\text{KH}}(\mathbf{r}, t) = e^{-\frac{i}{\hbar} \int^t dt' \frac{e\mathbf{A}(t') \cdot \mathbf{P}}{m_e}} \Psi(\mathbf{r}, t) \quad (4.24)$$

By direct calculation we get the equation obeyed by the wavefunction in the Kramers-Henneberger form

$$i\hbar \frac{\partial \Psi_{\text{KH}}(\mathbf{r}, t)}{\partial t} = \frac{\mathbf{P}^2}{2m_e} \Psi_{\text{KH}}(\mathbf{r}, t) + e^{-\frac{i}{\hbar} \int^t dt' \frac{e\mathbf{A}(t') \cdot \mathbf{P}}{m_e}} V(\mathbf{r}) e^{\frac{i}{\hbar} \int^t dt' \frac{e\mathbf{A}(t') \cdot \mathbf{P}}{m_e}} \psi_{\text{KH}}(\mathbf{r}, t) \quad (4.25)$$

The identity

$$e^{\frac{i}{\hbar} \mathbf{a} \cdot \mathbf{P}} V(\mathbf{r}) e^{-\frac{i}{\hbar} \mathbf{a} \cdot \mathbf{P}} = V(\mathbf{r} + \mathbf{a}) \quad (4.26)$$

allows us to write the Schrödinger equation as

$$i\hbar \frac{\partial \Psi_{\text{KH}}(\mathbf{r}, t)}{\partial t} = \frac{\mathbf{P}^2}{2m_e} \Psi_{\text{KH}}(\mathbf{r}, t) + V(\mathbf{r} - \int^t dt' \frac{e\mathbf{A}(t')}{m_e}) \Psi_{\text{KH}}(\mathbf{r}, t). \quad (4.27)$$

Usually the notation

$$\alpha(t) = -\frac{e}{m_e} \int^t dt' \mathbf{A}(t') \quad (4.28)$$

is introduced; $\alpha(t)$ is the classical trajectory of a particle of mass m_e and charge e in the field described by the potential $\mathbf{A}(t)$ (see Eq. (2.5)) and the Kramers-Henneberger form of the Schrödinger equation is

$$i\hbar \frac{\partial \Psi_{\text{KH}}(\mathbf{r}, t)}{\partial t} = \frac{\mathbf{P}^2}{2m_e} \Psi_{\text{KH}}(\mathbf{r}, t) + V(\mathbf{r} + \alpha(t)) \Psi_{\text{KH}}(\mathbf{r}, t). \quad (4.29)$$

It can be seen as obtained from the form (4.21) of the Schrödinger equation by performing a *spatial translation* by the classical trajectory $\alpha(t)$.

Chapter 5

The two-level system

5.1 The isolated two level system

We consider a two level system, with the nondegenerate energy levels $E_g < E_e$ and the eigenvectors $|g\rangle, |e\rangle$. The state space is then two-dimensional, the two energy eigenvectors forming a basis. We define the characteristic frequency ω_0 as

$$\omega_0 = \frac{E_e - E_g}{\hbar}. \quad (5.1)$$

The state vector of the system at any moment t can be written as

$$|\psi(t)\rangle = c_1 e^{-\frac{i}{\hbar} E_g t} |g\rangle + c_2 e^{-\frac{i}{\hbar} E_e t} |e\rangle, \quad |c_1|^2 + |c_2|^2 = 1, \quad (5.2)$$

where c_1, c_2 are constant complex numbers.

In this basis, any observable is represented by a hermitian 2×2 matrix

$$\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}, \quad A_{11} = A_{11}^*, \quad A_{22} = A_{22}^*. \quad (5.3)$$

Its expectation value is

$$\bar{\mathcal{A}} = A_{11} |c_1|^2 + A_{22} |c_2|^2 + (c_1^* c_2 A_{12} + c_1 c_2^* A_{12}^*) \cos \omega_0 t - i(c_1^* c_2 A_{12} - c_1 c_2^* A_{12}^*) \sin \omega_0 t, \quad (5.4)$$

a periodic function with the characteristic period of the system $T_0 = 2\pi/\omega_0$.

5.2 The two-level system in a static external field

The interaction Hamiltonian of the two level system with an external field is described by a 2×2 hermitian matrix; we shall assume this matrix is off diagonal, i.e. it has the form in the basis $|g\rangle, |e\rangle$

$$V = \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix}, \quad v - \text{complex number}. \quad (5.5)$$

We study the case when at the moment $t = 0$ the system is in the state $|g\rangle$. We solve the evolution equation in the *interaction picture*: the hamiltonian is $\hat{V}_I = \exp(\frac{i}{\hbar} \hat{H}_0 t) \hat{V} \exp(-\frac{i}{\hbar} \hat{H}_0 t)$. In the basis $|g\rangle, |e\rangle$ the matrix of the operator $\exp(\frac{i}{\hbar} \hat{H}_0 t)$ is diagonal

$$e^{\frac{i}{\hbar} \hat{H}_0 t} \rightarrow \begin{pmatrix} \exp(\frac{i}{\hbar} E_g t) & 0 \\ 0 & \exp(\frac{i}{\hbar} E_e t) \end{pmatrix}. \quad (5.6)$$

and the matrix of the Hamiltonian is

$$V_I = \begin{pmatrix} 0 & v e^{-i\omega_0 t} \\ v^* e^{i\omega_0 t} & 0 \end{pmatrix}, \quad (5.7)$$

with ω_0 defined in (5.1). We look for a solution of the Schrödinger equation as

$$|\Psi_I(t)\rangle = c_g(t) |g\rangle + c_e(t) |e\rangle. \quad (5.8)$$

which leads to the system of equations

$$i\hbar \frac{dc_g}{dt} = v e^{-i\omega_0 t} c_e(t), \quad i\hbar \frac{dc_e}{dt} = v^* e^{i\omega_0 t} c_g(t). \quad (5.9)$$

The previous system can be written as a second order equation for one of the two coefficients

$$\frac{d^2 c_g}{dt^2} + i\omega_0 \frac{dc_g}{dt} + \frac{|v|^2}{\hbar^2} c_g = 0. \quad (5.10)$$

For the previous equation we look for the solutions of the form

$$c_g(t) = \exp\left(-\frac{i}{\hbar} \lambda t\right), \quad (5.11)$$

where λ obeys the equation

$$\lambda^2 - \hbar\omega_0 \lambda - |v|^2 = 0. \quad (5.12)$$

The possible values of λ are

$$\lambda_1 = \hbar\left(\frac{\omega_0}{2} + \Omega\right), \quad \lambda_2 = \hbar\left(\frac{\omega_0}{2} - \Omega\right), \quad \Omega = \sqrt{\frac{\omega_0^2}{4} + \frac{|v|^2}{\hbar^2}}. \quad (5.13)$$

Then the general solution of the equation (5.10) is

$$c_g(t) = d_1 \exp\left(-\frac{i}{\hbar} \lambda_1 t\right) + d_2 \exp\left(-\frac{i}{\hbar} \lambda_2 t\right) = e^{-\frac{i}{2} \omega_0 t} (d_1 e^{-i\Omega t} + d_2 e^{i\Omega t}). \quad (5.14)$$

The coefficient $c_e(t)$ is obtained from the first equation of the system as

$$c_e(t) = i \frac{\hbar}{v} \frac{dc_g}{dt} e^{i\omega_0 t} = \frac{1}{v} e^{\frac{i}{2} \omega_0 t} (\lambda_1 d_1 e^{-i\Omega t} + \lambda_2 d_2 e^{i\Omega t}). \quad (5.15)$$

The initial condition $|\Psi_I(0)\rangle = |g\rangle$ gives $c_g(0) = 1$, $c_e(0) = 0$, i.e. $d_1 + d_2 = 1$ and $\lambda_1 d_1 + \lambda_2 d_2 = 0$. The constants d_1 are d_2

$$d_1 = \frac{1}{2} - \frac{\omega_0}{4\Omega}, \quad d_2 = \frac{1}{2} + \frac{\omega_0}{4\Omega} \quad (5.16)$$

with the final result

$$c_g(t) = e^{-\frac{i}{2} \omega_0 t} \left(\cos \Omega t + i \frac{\omega_0}{2\Omega} \sin \Omega t \right), \quad c_e(t) = -i \frac{v^*}{\hbar \Omega} e^{\frac{i}{2} \omega_0 t} \sin \Omega t. \quad (5.17)$$

The probability for the system to be in the ground, respectively the excited state

$$|c_g(t)|^2 = \cos^2 \Omega t + \frac{\omega_0^2}{4\Omega^2} \sin^2 \Omega t, \quad p_{\text{exc}} = \frac{|v|^2}{\hbar^2 \Omega^2} \sin^2 \Omega t. \quad (5.18)$$

is a periodic function with the period $T = \pi/\Omega$. The peak value of the transition probability tends to the unity when v increases.

When ω_0 is small the two energy levels are close, and for $\omega_0 \rightarrow 0$ we have $E_e = E_g$ (a degenerate energy level). In this case $\Omega = |v|/\hbar$, so the peak value of the excitation probability becomes

$$p_{\text{exc}}(t) = \sin^2\left(\frac{|v|}{\hbar} t\right). \quad (5.19)$$

At the moments $\tau_N = (2N+1)\pi\hbar/(2|v|)$, $N = 1, 2, \dots$ the excitation probability is unity, so the system oscillates between the two states with the frequency $|v|/\hbar$, named *resonance frequency* between the two degenerate states.

We consider now the time evolution of the same system, but for the initial state

$$|\Psi_0\rangle = \alpha_1 |g\rangle + \alpha_2 |e\rangle, \quad |\alpha_1|^2 + |\alpha_2|^2 = 1. \quad (5.20)$$

which leads to the modified initial conditions imposed to the state vector (5.8)

$$c_g(0) = \alpha_1, c_e(0) = \alpha_2 \rightarrow d_1 + d_2 = \alpha_1, \lambda_1 d_1 + \lambda_2 d_2 = v \alpha_2. \quad (5.21)$$

By direct calculation we find the coefficients d_1 and d_2 , and using them in (5.14),(5.15) we get

$$c_g(t) = \left[\alpha_1 \cos \Omega t + \frac{i}{2\Omega} (\alpha_1 \omega_0 - 2\alpha_2 \frac{v}{\hbar}) \sin \Omega t \right] e^{-\frac{i}{2} \omega_0 t}, \quad (5.22)$$

$$c_e(t) = \left[\alpha_2 \cos \Omega t - \frac{i}{2\Omega} (2\alpha_1 \frac{v^*}{\hbar} + \alpha_2 \omega_0) \sin \Omega t \right] e^{\frac{i}{2} \omega_0 t}. \quad (5.23)$$

As in the previous case, the probability to find the system in one of the states $|g\rangle, |e\rangle$ is a periodic function of time.

5.2.1 The 1/2 spin particle in rotating magnetic field

The particle with spin 1/2 at rest is the simplest two-level system. We shall consider this system using the previous notations $|g\rangle$ and $|e\rangle$ for the states with the spin projection on the third axis of the reference frame is 1/2 and respectively -1/2.

We assume that the particle has a magnetic moment $\mu = \mu \sigma$ and it is in the presence of a rotating magnetic field given by

$$B_x = \sin \beta \cos \omega t, B_y = \sin \beta \sin \omega t, B_z = \cos \beta. \quad (5.24)$$

Vectorul inducției magnetice se rotește continuu cu pulsația unghiulară ω în jurul axei z . The Hamiltonian matrix is

$$h(t) = -\mu \mathbf{B}(t) \cdot \sigma = -\mu B \begin{pmatrix} \cos \beta & \sin \beta e^{-i\omega t} \\ \sin \beta e^{i\omega t} & -\cos \beta \end{pmatrix}. \quad (5.25)$$

The evolution equation of the state vector written as

$$|\Psi(t)\rangle = a_1(t) |g\rangle + a_2(t) |e\rangle \quad (5.26)$$

becomes a system of two coupled equations for the coefficients $a_1(t)$ and $a_2(t)$.

$$\frac{da_1}{dt} = i\omega_L (a_1 \cos \beta + a_2 e^{-i\omega t} \sin \beta), \quad (5.27)$$

$$\frac{da_2}{dt} = i\omega_L (a_1 e^{i\omega t} \sin \beta - a_2 \cos \beta), \quad \omega_L = \frac{\mu B}{\hbar}, \quad (5.28)$$

We look for a solution as

$$a_1(t) = c_1 e^{-i\lambda t}, a_2(t) = c_2 e^{-i(\lambda-\omega)t}, \quad (5.29)$$

With the expressions (5.29), the equation system (5.27) becomes a linear and homogeneous system for the coefficients

$$(\lambda + \omega_L \cos \beta) c_1 + \omega_L \sin \beta c_2 = 0, \quad (5.30)$$

$$\omega_L \sin \beta c_1 + (\lambda - \omega - \omega_L \cos \beta) c_2 = 0, \quad (5.31)$$

The system has a non-vanishing solution if λ is a solution of the equation

$$\lambda^2 - \omega \lambda - \omega_L (\omega \cos \beta + \omega_L) = 0. \quad (5.32)$$

The two roots of the above equation are

$$\lambda_{a,b} = \frac{\omega}{2} \mp \Delta, \Delta = \sqrt{\frac{\omega^2}{4} + \omega \omega_L \cos \beta + \omega_L^2}. \quad (5.33)$$

The the coefficient a_1 can be written as

$$a_1(t) = C_1 e^{-i\lambda_a t} + C_2 e^{-i\lambda_b t}, \quad (5.34)$$

with C_1 and C_2 constants. The other component $a_2(t)$ is now determined by the first equation of (5.27),

$$a_2(t) = -\frac{e^{i\omega t}}{\sin \beta} \left[(\cos \beta + \frac{\lambda_a}{\omega_L}) C_1 e^{-i\lambda_a t} + (\cos \beta + \frac{\lambda_b}{\omega_L}) C_2 e^{-i\lambda_b t} \right]. \quad (5.35)$$

As an example, for the initial condition $a_1(0) = 1$, $a_2(0) = 0$ we get the state vector as

$$\begin{aligned} |\Psi(t)\rangle = & e^{-i\omega/2 t} \left[\cos \Delta t + \frac{i}{\Delta} \left(\frac{\omega}{2} + \omega_L \cos \beta \right) \sin \Delta t \right] |g\rangle \\ & + i \omega_L e^{i\omega/2 t} \frac{\sin \beta}{\Delta} \sin \Delta t |e\rangle. \end{aligned} \quad (5.36)$$

and the probability to find the system in the excited state at an arbitrary moment t

$$p_{g \rightarrow e}(t) = \left(\frac{\omega_L}{\Delta} \right)^2 \sin^2 \beta \sin^2 \Delta t. \quad (5.37)$$

For $\beta = 0$ the magnetic field is static, along Oz . Then the Hamiltonian is constant, the vectors $|g\rangle$, $|e\rangle$ are its eigenvectors for the eigenvalues $-\hbar\omega_L$ and $\hbar\omega_L$. The solution (5.36) reduces to

$$|\Psi_{st}(t)\rangle = e^{i\omega_L t} |g\rangle. \quad (5.38)$$

5.3 The two level atom with a harmonic perturbation. The rotating wave approximation

We consider the two level system, in the ground state at $t = 0$ interacting with

$$V(t) = \begin{pmatrix} 0 & \mathcal{A}_0 \cos(\omega t + \phi_0) \\ \mathcal{A}_0^* \cos(\omega t + \phi_0) & 0 \end{pmatrix}. \quad (5.39)$$

Using the previously defined notations, in the interaction picture, the components of the state vector in the basis of the energy eigenvectors $|g\rangle$, $|e\rangle$, denoted by $c_g(t)$, $c_e(t)$, obey the equations

$$i\hbar \frac{dc_g}{dt} = e^{-i\omega_0 t} \mathcal{A}_0 \cos(\omega t + \phi_0) c_e(t), \quad (5.40)$$

$$i\hbar \frac{dc_e}{dt} = \mathcal{A}_0^* e^{i\omega_0 t} \cos(\omega t + \phi_0) c_g(t), \quad (5.41)$$

with the initial conditions $c_g(0) = 1$, $c_e(0) = 0$. The previous system has an exact solution. We notice that the system coefficients are oscillatory functions of $(-\omega_0 \pm \omega)t$, $(\omega_0 \pm \omega)t$. We shall neglect the fast-oscillating terms, and keep only the terms containing the exponents $\pm(\omega - \omega_0)$. The approximation is valid if $|\omega_0 - \omega| \ll \omega$; in this case the system becomes

$$i\hbar \frac{dc_g}{dt} = \frac{\mathcal{A}_0}{2} e^{i\phi_0} e^{i(-\omega_0 + \omega)t} c_e(t), \quad (5.42)$$

$$i\hbar \frac{dc_e}{dt} = \frac{\mathcal{A}_0^*}{2} e^{-i\phi_0} e^{i(\omega_0 - \omega)t} c_g(t). \quad (5.43)$$

The above system is identical with (5.9) and we could write its solution, with the correspondence of notations

$$\omega_0 \rightarrow -\delta\omega \equiv -(\omega - \omega_0), \quad v \rightarrow \frac{\mathcal{A}_0}{2} e^{i\phi_0}. \quad (5.44)$$

This leads to

$$\Omega = \frac{1}{2} \omega_R, \quad \omega_R = \sqrt{(\delta\omega)^2 + \frac{|\mathcal{A}_0|^2}{\hbar^2}}. \quad (5.45)$$

ω_R is named Rabi frequency.

The excitation probability, obtained from (5.18) with the notations indicated above is

$$p_{\text{exc}}(t) = \frac{|\mathcal{A}_0|^2}{\hbar^2 \omega_R^2} \sin^2 \frac{\omega_R t}{2}. \quad (5.46)$$