Divided differences for a function $f(x)$

The divided differences are defined recursively as:

$$f[x_i] = f(x_i) \tag{1}$$
$$f[x_{i-1}, x_i] = \frac{f[x_i] - f[x_{i-1}]}{x_i - x_{i-1}} \tag{2}$$
$$f[x_{i-2}, x_{i-1}, x_i] = \frac{f[x_{i-1}, x_i] - f[x_{i-2}, x_{i-1}]}{x_i - x_{i-2}} \tag{3}$$
$$\vdots$$
$$f[x_{i-j}, x_{i-j+1}, \ldots, x_i] = \frac{f[x_{i-j+1}, x_{i-j+2}, \ldots, x_i] - f[x_{i-j}, x_{i-j+1}, \ldots, x_{i-1}]}{x_i - x_{i-j}}. \tag{4}$$

x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$\ldots$	$f[x_{i-j}, x_{i-j+1}, \ldots, x_i]$
x_0	$f[x_0]$				
x_1	$f[x_1]$	$f[x_0, x_1]$			
x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$		
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_j	$f[x_j]$	$f[x_{j-1}, x_j]$	$f[x_{j-2}, x_{j-1}, x_j]$	$\ldots$	$f[x_0, x_1, \ldots, x_j]$

Table 1: Divided differences table.

Newton Interpolation

Newton Interpolation is one of the polynomial methods used for the interpolation of a data set of points. It uses divided differences.

We consider known a set of $n+1$ points $(x_i, y_i),\ i = 0, 1, 2, \ldots, n$:

$$\begin{array}{ll} x_0 & y_0 = f(x_0) \\ \vee & \\ x_1 & y_1 = f(x_1) \\ \vee & \\ x_2 & y_2 = f(x_2) \\ \vee & \\ \vdots & \vdots \\ \vee & \\ x_n & y_n = f(x_n), \end{array} \tag{5}$$

but not the analytic expression of the function $f(x)$. The goal is to estimate $f(x)$ at an arbitrary x using smooth curves through and beyond all x_i, for $i = 0, 1, 2, \dots, n$. By *interpolation*, we understand the estimation of $f(x)$ for any $x \in [x_0, x_n]$. The *extrapolation* is the estimation of $f(x)$ for $x \in (-\infty, x_0] \cup [x_n, \infty)$.[1]

The *order of interpolation* is given by the number of points used in the interpolation scheme minus one, i.e. $n + 1 - 1 = n$.

The Newton interpolation is based on the Newton polynomial approximation that can be derived like an expansion, but based on multiple centers $(x_0, x_1, \dots, x_n)$.

Theorem (Newton polynomial)

Assume $f \in C^{n+1}[a, b]$ and the set of $n + 1$ distinct points

$$(x_i, y_i),\ \ i = 0, 1, 2, \dots, n,\ \text{ with } x_i \in [a, b] \text{ and } y_i = f(x_i). \tag{6}$$

Then $f(x)$ can be written as

$$f(x) = P_n(x) + R_n(x),$$

where $P_n(x)$ is a polynomial of degree n that can be used to approximate $f(x)$ and is given by

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\dots(x - x_{n-1}). \tag{7}$$

The coefficients $a_i,\ i = 0, 1, 2, \dots, n$ are constructed using divided differences.

As $f(x) \approx P_n(x)$, we have for the given set of points:

$$y_i = f(x_i) = P_n(x_i) \tag{8}$$

and the polynomial goes through the $n + 1$ points $(x_i, y_i),\ i = 0, 1, 2, \dots, n$.

The remainder is given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - x_0)(x - x_1)\dots(x - x_n),$$

for c a value in the interval $[a, b]$.

The coefficient a_i of the Newton polynomial (7) is

$$a_i = f[x_0, x_1, \dots, x_i], \tag{9}$$

i.e. the most to the right element on the row corresponding to the current x_i in table 1.

For convenience, let us denote

$$d_{i,j} = f[x_{i-j}, x_{i-j+1}, \dots, x_i]. \tag{10}$$

Then

$$d_{i,0} = f[x_i],\ \ i = 0, 1, \dots, n \tag{11}$$

$$d_{i,j} = \frac{d_{i,j-1} - d_{i-1,j-1}}{x_i - x_{i-j}},\quad \begin{array}{l} i = 1, 2, \dots, n \\ j = 1, 2 \dots, i. \end{array} \tag{12}$$

[1] Note that interpolation and extrapolation differ from function approximation. For the first two $f(x_i)$ is not given at points on our choice, while in the function approximation one tries to find an approximate easy to compute function.

With the above notations, the Newton polynomial is

$$P_n(x) = d_{0,0} + d_{1,1}(x - x_0) + d_{2,2}(x - x_0)(x - x_1) + \cdots + d_{n,n}(x - x_0)(x - x_1)\ldots(x - x_{n-1}). \tag{13}$$

and can be created recursively:

$$P_0(x) = d_{0,0} \tag{14}$$

$$P_1(x) = P_0(x) + d_{1,1}(x - x_0) \tag{15}$$

$$P_2(x) = P_1(x) + d_{2,2}(x - x_0)(x - x_1) \tag{16}$$

$$\vdots$$

$$P_n(x) = P_{n-1}(x) + d_{n,n}\prod_{i=1}^{n-1}(x - x_i). \tag{17}$$

Algorithm

- Give the list of points:[2]

$$\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}. \tag{18}$$

- Calculate the divided differences using the known values $y_i = f(x_i)$.
  - For $i = 1, 2, \ldots, n,\ d_{i,1} = y_i$.
  - For $i = 1, 2, \ldots, n-1$, and for $j = 1, 2, \ldots, i$,

$$d_{i+1,j+1} = \frac{d_{i+1,j} - d_{i,j}}{x_{i+1} - x_{i-j+1}}. \tag{19}$$

- Calculate the Newton polynomial of degree $n-1$:

$$P(x) = d_{1,1} + \sum_{i=1}^{n-1}\left(d_{i+1,i+1}\prod_{j=1}^{i}(x - x_j)\right) \quad \text{or} \quad P(x) = d_{1,1} + \sum_{i=2}^{n}\left(d_{i,i}\prod_{j=1}^{i-1}(x - x_j)\right). \tag{20}$$

[2] In order to implement the set of points as a Mathematica list, we start with (x_1, y_1) instead of (x_0, y_0), having thus n points instead of $n+1$.