VIII. Numerical Solution of Partial Differential Equations:
Finite Difference Methods for PDE of order II

- a complex topic; separate text books
- ground of computer analysis and simulation in physics mathematics, biology, ...

PDE: $$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} = f\left(x,y,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$$

define $d = b^2 - ac$

Clasification: 1. Elliptic PDE $d<0$
2. $d=0$ Parabolic PDE
3. $d>0$ Hyperbolic PDE

VIII.1. Elliptic Partial Differential Equations

e.g. Laplace equation: $\Delta u(x,y) = 0 \iff \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
Poisson equation: $\Delta u = f(x,y)$
Helmholtz equation: $\Delta u + k \cdot u = f(x,y)$

$x \in \Omega \subset \mathbb{R}^m$ - open, finite set with (regular/smooth boundaries

Boundary conditions:

a) Dirichlet b.c. - specify the value of the function on the surface: $u\Big|_{\partial\Omega}$

b) Neumann b.c. - -//- normal derivative of the func -//- -//- $\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$ ν - versorul normalei exterioare

c) Robin -//-

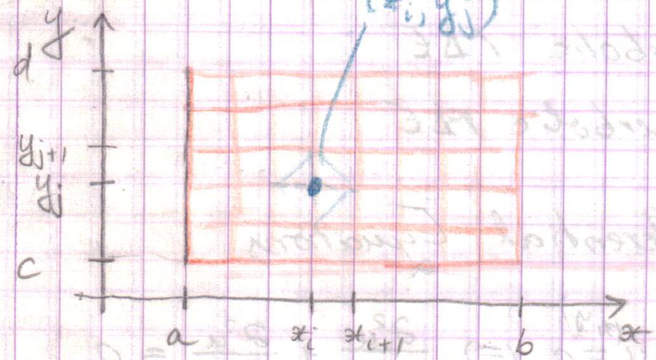
Example: Dirichlet problem for the Poisson equation in an rectangular domain

$$(1)\begin{cases}\Delta u = f & \text{in } \Omega=(a,b)\times(c,d) & (1.1)\\ u\big|_{\partial\Omega}=u_0 & & (1.2)\end{cases}$$

Let be $h=\frac{b-a}{m}$ and $k=\frac{d-c}{n}$ and take

$$x_{i+1}=x_i+h \quad ; \quad i=0,1,2,\dots,m-1$$

$$y_{j+1}=y_j+k \quad , \quad j=0,1,2,\dots,n-1$$



notation: $u(x_i,y_i)=u_{i,j}$ $\quad i=\overline{0,m}$ and $j=\overline{0,n}$

and $f(x_i,y_i)=f_{i,j}$ $\quad i=\overline{1,m-1}$ and $j=\overline{1,n-1}$

- using the 3 point rule for derivation, one can obtain:

$$(2)\begin{cases}\frac{\partial u}{\partial x}(x_i,y_i)\simeq\frac{u_{i+1,j}-u_{i-1,j}}{2h}\\ \frac{\partial^2 u}{\partial x^2}(x_i,y_i)\simeq\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2} \quad \text{and similarly}\\ \frac{\partial u}{\partial y}(x_i,y_i)\simeq\frac{u_{i,j+1}-u_{i,j-1}}{2k}\\ \frac{\partial^2 u}{\partial y^2}(x_i,y_i)\simeq\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2}\end{cases}$$

(1,2)

$\Rightarrow$ (1.1) for (x_i, y_j) $i=\overline{1,m-1}$ and $j=\overline{1,n-1}$

and (1.2) for (x_0, y_j) and (x_m, y_j) $j=\overline{0,n}$

(x_i, y_0) and (x_i, y_n) $i=\overline{0,m}$

$$\Leftrightarrow \begin{cases} \dfrac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2} + \dfrac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2} = f_{i,j} & i=\overline{1,m-1} \\ & j=\overline{1,n-1} \\ & (3.1) \\ \text{+ the values } u_{0,j};\ u_{m,j};\ j=\overline{0,n} & \text{known} \\ u_{i,0};\ u_{i,n};\ i=\overline{0,m} \end{cases} \quad (3)$$

$$(3.1) \Leftrightarrow k^2(u_{i+1,j}+u_{i-1,j}) + h^2(u_{i,j+1}+u_{i,j-1}) - 2u_{i,j}(k^2+h^2) = f_{i,j}\cdot k^2 h^2 \quad \Big/ \frac{1}{k^2}$$

using $r=\frac{k}{h}$

n multiply 3.1 by h^2

$$(3.1) \Leftrightarrow r^2(u_{i+1,j}+u_{i-1,j}) + u_{i,j+1} + u_{i,j-1} - 2u_{i,j}(r^2+1) = f_{i,j}\, r^2 \cdot h^2 \quad (4)$$

$i=\overline{1,m-1}$; $j=\overline{1,n-1}$

$: \quad -2(r^2+1)u_{1,1} + r^2 u_{2,1} + u_{1,2} = f_{1,1} r^2 h^2 - r^2 u_{0,1} - u_{1,0}$

$1: \quad r^2 \cdot u_{1,1} - 2(r^2+1)u_{2,1} + r^2 u_{3,1} + u_{2,2} = f_{2,1} r^2 h^2 - u_{2,0}$

$1: \quad 0 + r^2 u_{2,1} - 2(r^2+1)u_{3,1} + r^2 u_{4,1} + u_{3,2} = f_{3,1} r^2 h^2 - u_{3,0}$

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$1: \quad r^2 u_{m-2,1} - 2(r^2+1)u_{m-1,1} + u_{m-1,2} = f_{m-1,1} r^2 h^2 - u_{m,1} r^2 - u_{m-1,0}$

$2: \quad -2(r^2+1)u_{1,2} + u_{1,3} + u_{1,1} + r^2 u_{2,2} = f_{1,2} r^2 h^2 - r^2 u_{0,2}$

$n-1: \quad -2(r^2+1)u_{m-1,n-1} + r^2 u_{m-2,n-1} + u_{m-1,n-2} = f_{m-1,n-1} - r^2 u_{m,n-1} - u_{m-1,n}$

the system $A \cdot X = B$

denote:

$$X = \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ \vdots \\ u_{m-1,1} \\ u_{1,2} \\ u_{2,2} \\ \vdots \\ u_{m-1,2} \\ u_{1,3} \\ \vdots \\ u_{m-1,3} \\ \vdots \\ u_{1,n-1} \\ \vdots \\ u_{m-1,n-1} \end{pmatrix} \qquad B = \begin{pmatrix} r^2h^2 f_{1,1} - u_{1,0} - r^2 u_{0,1} \\ r^2h^2 f_{2,1} - u_{2,0} \\ \vdots \\ r^2h^2 f_{i,1} - u_{i,0} \\ r^2h^2 f_{m-1,1} - u_{m-1,0} - r^2 u_{m,1} \end{pmatrix}$$

$$A = \begin{pmatrix} A_1 & I_{m-1} & 0 & - & - & - & 0 \\ I_{m-1} & A_2 & I_{m-1} & 0 & - & - & 0 \\ 0 & I_{m-1} & A_3 & I_{m-1} & 0 & - & 0 \\ - & - & - & - & - & - & - \\ 0 & - & - & I_{m-1} & A_{n-2} & I_{m-1} \\ 0 & - & - & - & I_{m-1} & A_{n-1} \end{pmatrix}$$

where $A_1 = A_2 = \dots = A_{n-1} = \begin{pmatrix} -2(r^2+1) & r^2 & 0 & - & - & 0 & 0 \\ r^2 & -2(r^2+1) & r^2 & - & - & 0 & 0 \\ 0 & r^2 & -2(r^2+1) & - & - & 0 & 0 \\ - & - & - & - & - & - & - \\ 0 & 0 & 0 & \dots & & r^2 & -2(r^2+1) \end{pmatrix}$

$\in \mathcal{M}_{m-1}\ (m-1)\times(m-1)$

I_{m-1} – unit matrix of order $(m-1)\times(m-1)$

0 – nul matrix

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Remark:

if $r=1$ and $f=0$, then eq. (4) becomes

$$u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{4} \qquad i=\overline{1,m-1},\ j=\overline{1,n-1}$$

i.e. the value of u in any point inside the rectangular equals the arithmetic mean of the values in neighbouring points (nodari). This property is analogous to the one in harmonic functions theory saying that the value of an harmonic function in a point equals the mean of the values of the functions on a sphere contained in the domain Ω.

Due to the large dimension of the A matrix and ~~due~~ to its diagonally dominant character, one prefers solving the system (4) with the conditions (3,2) using iterative methods:

a) Jacobi than transf to G-S — direct din ec (4)

b) Gauss-Seidel: (5) $x_i^{(k)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)}\right)$

$$(4) \Rightarrow \boxed{u_{i,j}^{(k)} = \frac{r^2}{2(r^2+1)}\left\{ u_{i+1,j}^{(k-1)} + u_{i-1,j}^{(k)} + \frac{1}{r^2}\left(u_{i,j+1}^{(k-1)} + u_{i,j-1}^{(k)}\right) - h^2 f_{i,j}\right\}} \qquad (6)$$

$$k=1,2,\dots \qquad i=\overline{1,m-1},\ j=\overline{1,n-1}$$

as above

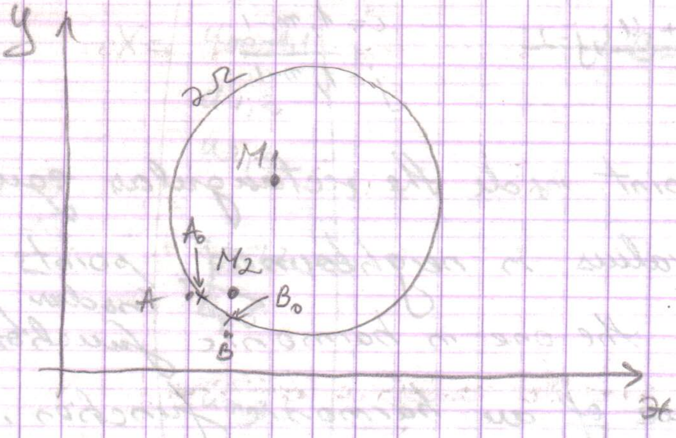
c) SOR: $$\boxed{u_{i,j}^{(k)} = \omega \cdot \bar{u}_{i,j} + (1-\omega)\, u_{i,j}^{(k-1)}} \qquad \text{—''—} \qquad (7)$$

– the weighted average ω can be chosen in an optimal way as:

$$\omega_{opt} = \frac{4\left(2-\sqrt{4-t^2}\right)}{t^2}, \text{ where } t = \cos\left(\frac{\pi}{m}\right) + \cos\left(\frac{\pi}{r\cdot n}\right)$$

$$= \frac{4(4-4+t^2)}{t^2\left(2+\sqrt{4-t^2}\right)} = \frac{4t^2}{t^2\left(2+\sqrt{4-t^2}\right)}$$

Ω - simple domain, smooth curve $\partial\Omega$



- for points inside $\partial\Omega$ (M_1), apply directly eq (4)

- for points having at least one neighbouring point outside Ω or on $\partial\Omega$ (M_2), one uses the same approximating formulas but the points in the exterior are replaced by the points obtained from the intersection of $\partial\Omega$ with the net.

i.e. instead of A and B, we take A_0 and B_0