VII.2. Parabolic Partial Differential Equations

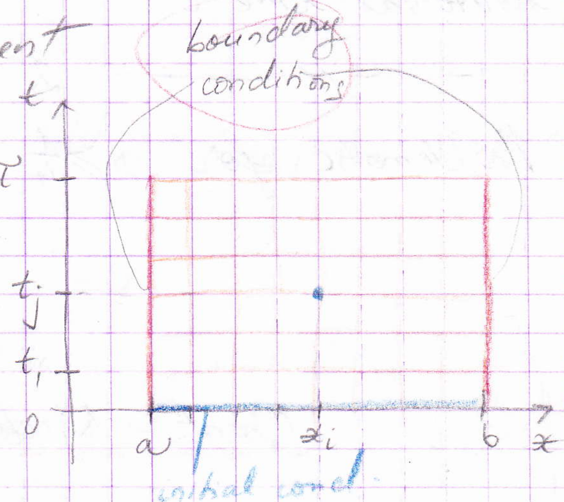
As example, consider the one-dimensional heat equation:

$$(8)\quad \begin{cases} \dfrac{\partial u}{\partial t} - c^2 \dfrac{\partial^2 u}{\partial x^2} = f(x,t) & x\in(a,b),\ t\in(0,\tau) \\ u\big|_{t=0} = \varphi(x) & \text{(initial condition)} \\ u\big|_{x=a} = \psi_1(t) \quad u\big|_{x=b} = \psi_2(t) & \text{(boundary conditions)} \end{cases}$$

The equation models the propagation of the heat through an insulated rod with negligible section

$c>0$ - termical conductivity coefficient

u - temperature field



Let be $h = \dfrac{b-a}{m}$; $\left(k = \dfrac{\tau}{n}\right)$ take (nespecificat)

$x_{i+1} = x_i + h, \quad i = 0,1,2,\dots,m-1$

$t_j = kj, \quad j = 0,1,2,\dots$

notation:

$u(x_i, t_j) = u_{i,j} \quad i=\overline{0,m},\ j=0,1,\dots \qquad \psi_1(t_j) = \psi_{1j} \quad j=1,2,\dots$

$\varphi(x_i) = \varphi_i \quad i=\overline{1,m-1} \qquad \psi_2(t_j) = \psi_{2j}$

$f(x_i, t_j) = f_{i,j}$

analogously to (3), from equations (2)

simple

$$(9)\quad \begin{cases} \dfrac{u_{i,j+1} - u_{i,j}}{k} - c^2 \dfrac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = f_{i,j} & i=\overline{1,m-1},\ j=1,2,\dots \quad (9.1) \\ u_{i,0} = \varphi_i \quad \text{(IC)} & (9.2) \\ u_{0,j} = \psi_{1j},\ u_{m,j} = \psi_{2j} \quad \text{(BC)} & (9.3) \end{cases}$$

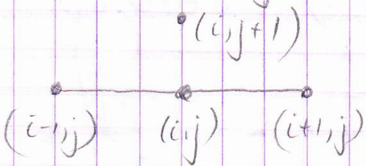
def. of deriv

take $f_{i,j} = 0$

$$(9.1) \Leftrightarrow h^2(u_{i,j+1} - u_{i,j}) - c^2 k\,(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) = f_{i,j} \cdot h^2 k$$

denote $\quad r = \dfrac{c^2 k}{h^2}$

$$(10) \quad \boxed{u_{i,j+1} = r\,(u_{i+1,j} + u_{i-1,j}) + (1-2r)\,u_{ij} + f_{ij}\, h^2 \cdot \frac{r}{c^2}}$$



eq. (10) offers an explicit scheme:

– the values of the function on the horizontal line 1 is expressed as a function of the known values of the function u on the horizontal line 0

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Problematic for $r \geq \frac{1}{4}$ $\Rightarrow$ the scheme becomes unstable.

Crank – Nicolson Method

in (8) one can approximate:

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) \simeq \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}(x_i, t_j) + \frac{\partial^2 u}{\partial x^2}(x_i, t_{j+1})\right)$$

$$\simeq \frac{1}{2}\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}\right)$$

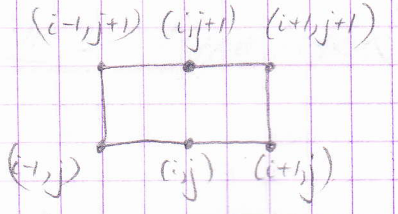
$\Rightarrow$ equation (8), in its discrete form is:

$\frac{1}{2}(2(1-2r)$

$$(11) \quad \boxed{-r\,(u_{i+1,j+1} + u_{i-1,j+1}) + 2(1+r)\,u_{i,j+1} = r\,(u_{i+1,j} + u_{i-1,j}) + 2(1-r)\,u_{ij} + 2\frac{h^2 r}{c^2} f_{ij}}$$

$i = \overline{1, m-1}$

$j = 1, 2, \ldots$



(11) is an implicit scheme. For a given j, 3 unknowns:

$$u_{i-1,j+1};\ u_{i,j+1},\ u_{i+1,j+1}$$

are calulated as a function of the values $u_{i-1,j}$; $u_{i,j}$; $u_{i+1,j}$ from the lower diagonal.

$$\begin{cases}
i=1: & 2(1+r)u_{1,j+1} - r\,u_{2,j+1} = 2(1-r)u_{1,j} + r\,u_{2,j} + r\,u_{0,j} + r\,u_{0,j+1} \\
i=2: & -r\,u_{1,j+1} + 2(1+r)u_{2,j+1} - r\,u_{3,j+1} = r\,u_{1,j} + 2(1-r)u_{2,j} + r\,u_{3,j} \\
i=3: & 0 - r\,u_{2,j+1} + 2(1+r)u_{3,j+1} - r \dots = \dots + \dots \\
\dots \\
i=m-1: & 0 - r\,u_{m-2,j+1} + 2(1+r)u_{m-1,j+1} = r(u_{m,j} + u_{m-2,j} + u_{m,j+1}) + 2(1-r)u_{m-1,j}
\end{cases}$$

$\Rightarrow$ matrix form of the system: $A \cdot X = B$

$$X = \begin{pmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{m-1,j+1} \end{pmatrix} \qquad B = \begin{pmatrix} r(u_{0,j+1} + u_{0,j} + u_{2,j}) + 2(1-r)u_{1,j} \\ r(u_{1,j} + u_{3,j}) + 2(1-r)u_{2,j} \\ \dots \\ r(u_{m,j+1} + u_{m,j} + u_{m-2,j}) + 2(1-r)u_{m-1,j} \end{pmatrix} + 2\frac{h^2 r}{c^2} f_{i,j}$$

$$A = \begin{pmatrix} 2(1+r) & -r & 0 & 0 & - & - & 0 & 0 \\ -r & 2(1+r) & -r & 0 & - & - & 0 & 0 \\ 0 & -r & 2(1+r) & -r & - & - & 0 & 0 \\ - & - & & & & & & \\ 0 & 0 & 0 & 0 & - & & -r & 2(1+r) \end{pmatrix}$$

A – tridiagonal, diagonally dominant matrix

$\Rightarrow$ one can use Gauss – Seidel to solve the system

a)
$$u^{(k)}_{i,j+1} = \frac{r}{2(1+r)}\left(u^{(k-1)}_{i+1,j+1} + u^{(k)}_{i-1,j+1} + \underbrace{u^{(k-1)}_{i+1,j} + u^{(k)}_{i-1,j} + \frac{2(1-r)}{r} \cdot u^{(k-1)}_{i,j} + 2\frac{h^2 r}{c^2} f_{i,j}}_{b_{i,j} \text{ contains } u_{i,j};\ \varphi_i,\ \psi_{1j},\ \psi_{2j} \text{ known}}\right) \quad i = 1, 2, \dots, m-1 \qquad (12)$$

the Crank–Nicolson is stable for any $r>0$

b) SOR:

$$u_{i,j+1}^{(k)} = \omega \cdot \underbrace{\left(\frac{r}{2(1+r)} \cdots \right)}_{\overline{u}_{i,j}} + (1-\omega)\, u_{i,j+1}^{(k-1)}$$

(13)

and $\omega_{opt} = \frac{2}{1+\sqrt{1-t^2}}$ $\quad t = \frac{r}{1+r} \cos\left(\frac{\pi}{m}\right)$