VII.3. Hyperbolic Partiall Differential Equations

- As an example consider the problem of the vibrant chord (problema coardei vibrante)

$$(14)\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = f(x,t) & x\in(a,b);\ t>0 \\ u\Big|_{t=0} = \varphi_1(x) \quad \dfrac{\partial u}{\partial t}\Big|_{t=0} = \varphi_2(x) & \text{(initial condition)} \\ u\Big|_{x=a} = \psi_1(t) \quad u\Big|_{x=b} = \psi_2(t) & \text{(boundary conditions)} \\ \psi_1(0) = \varphi_1(a);\ \psi_2(0) = \varphi_1(b) & \text{(connecting conditions)} \end{cases}$$

condiții de racordare

Analogously to parabolic equations

$$h = \frac{b-a}{m}, \qquad x_{i+1} = x_i + h,\ i = 0,1,2,\dots,m-1$$

$$t_j = k\,j,\ j = 0,1,2,\dots$$

+ notations:

$$u(x_i,t_j) = u_{i,j} \quad i=\overline{0,m},\ j=0,1,2,\dots \qquad f(x_i,t_j) = f_{i,j} \quad i=\overline{1,m-1},\ j=1,2,\dots$$

$$\varphi_1(x_i) = \varphi_{1i},\quad \varphi_2(x_i) = \varphi_{2i} \quad i=\overline{1,m-1} \qquad \psi_1(t_j) = \psi_{1j},\quad \psi_2(t_j) = \psi_{2j} \quad j=1,2,\dots$$

using the 3-point rule

$$(14) \Leftrightarrow (15) \begin{cases} \dfrac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} - c^2\,\dfrac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = f_{i,j} & i=\overline{1,m-1},\ j=1,2,\dots \\ u_{0,j} = \psi_{1j} \quad u_{m,j} = \psi_{2j} & j=1,2,\dots \\ u_{i,0} = \varphi_{1i} & i=\overline{1,m-1} \end{cases}$$

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$$u_{i,1} = u(x_i, k\cdot 1) \simeq \underbrace{u(x_i,0)}_{\varphi_1(x_i,0)} + k\underbrace{\frac{\partial u}{\partial t}(x_i,0)}_{\varphi_2(x_i,0)} + \frac{k^2}{2}\underbrace{\frac{\partial^2 u}{\partial t^2}(x_i,0)}_{c^2 \left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_i,0)} + f(x_i,0)\ (1)} + O(k^3)$$

$$= \varphi_{1i} + k\varphi_{2i} + \frac{k^2}{2}\left(c^2 \frac{u_{i+1,0} - 2u_{i,0} + u_{i-1,0}}{h^2} + f_{i,0}\right) + O(k^3)$$

$$= \varphi_{1i} + k\varphi_{2i} + \frac{k^2}{2}\left(c^2 \frac{\varphi_{1i+1} - 2\varphi_{1i} + \varphi_{1i-1}}{h^2} + f_{i,0}\right)$$

$$i = \overline{1, m-1}$$

$$(15.1) \Leftrightarrow c^2 \cdot \frac{k^2}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) - \left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) + k^2 f_{i,j} = 0$$

notation: $r = c^2 \frac{k^2}{h^2}$

$\Leftrightarrow$

$$\boxed{u_{i,j+1} = r\left(u_{i+1,j} + u_{i-1,j}\right) + 2(1-r)u_{i,j} - u_{i,j-1} + k^2 f_{i,j}} \quad (16)$$

$$i = \overline{1, m-1}\ ;\ j = 1, 2, \ldots$$

(16) is an explicit scheme envolving

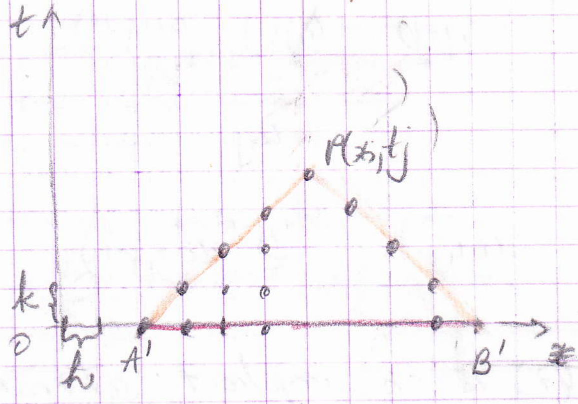
$$\begin{array}{ccc} & \bullet\,(i,j+1) & \\ \bullet\,(i-1,j) & \bullet\,(i,j) & \bullet\,(i+1,j) \\ & \bullet\,(i,j-1) & \end{array}$$

The values of the u function on the horizontal line 2 are expressed as a function of the known values of the u function on the horizontals 0 and 1

—//— hor. 3 —//— 1 and 2 and so on.

the scheme is stable $\Leftrightarrow$ $r \leq 1$

For the case: $r=1$ and $f=0$, if the EXACT values of the function u on the first 2 horizontals are known, than eq. (16) gives the exact solution in all points (i,j) (nodes).

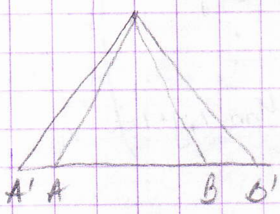


The line segment AB is called DEPENDENCE DOMAIN of the solution, i.e. the solution in P is given only by the initial conditions on the segment AB.

The numerical solution in $P(x_i, t_j)$ is given only by the initial conditions on the segment A'B'

The slope of PA' is $\frac{k}{h} = \frac{\sqrt{r}}{c}$

— " — PA is $\frac{1}{c}$



For a stable scheme it is necessary that ΔPAB is contained in $\Delta PA'B'$ (a modification of the initial condition on AA' / BB' does not affect the solution in P)

$$\Rightarrow \frac{\sqrt{r}}{c} \leq \frac{1}{c} \iff r \leq 1$$

ca să fie mai puțin abruptă panta

Crank-Nicolson Method

As for parabolic eq, one can approximate

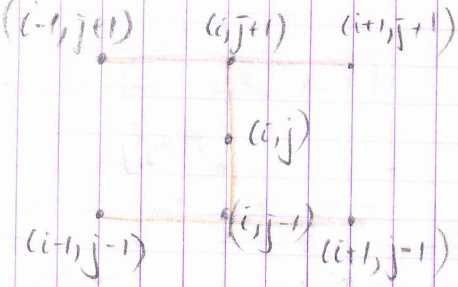
$$\frac{\partial^2 u}{\partial x^2}(x_i; t_j) \simeq \frac{1}{2}\left(\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} + \frac{u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}}{h^2} \right)$$

$\Rightarrow$ equation (14), in its discrete form is:

$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} - \frac{r}{2}\left(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}\right) = k^2 f_{ij}$$

$$\iff \boxed{-r\left(u_{i+1,j+1} + u_{i-1,j+1}\right) + 2(1+r)\,u_{i,j+1} = r\left(u_{i+1,j-1} + u_{i-1,j-1} - 2u_{i,j-1}\right) - 2u_{i,j-1} + 4u_{i,j} + 2k^2 f_{ij}} \quad (17)$$

$i = \overline{1, m-1}$, $j = 1, 2, \dots$



(17) is an implicit scheme. For a given j, 3 unknowns $u_{i-1,j+1}$, $u_{i,j+1}$, $u_{i+1,j+1}$ are determined as a function of the values from the previously 2 horizontal lines.

The linear system in (17) reduces to:

$A \cdot X = B$ where

$$X = \begin{pmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{m-1,j+1} \end{pmatrix} \qquad B = \begin{pmatrix} r(u_{0,j-1} + u_{1,j-1} - 2u_{1,j}) - 2u_{1,j-1} + 4u_{1,j} + 2k^2 f_{1,j} + r u_{0,j} \\ \text{general} \; — \\ r(u_{m,j-1} + u_{m-2,j-1} - 2u_{m-1,j-1}) - 2u_{m,j-1} + 4u_{m-1,j} + 2k^2 f_{m-1,j} + r u_{m,j+1} \end{pmatrix}$$

$$A = \text{Tridiag}(-r, 2(1+r), -r)$$

a) Gauss-Seidel:

$$u_{i,j+1}^{(k)} = \frac{r}{2(1+r)}\left(u_{i-1,j+1}^{(k)} + u_{i+1,j+1}^{(k-1)}\right) + b_{i,j}\frac{1}{2(1+r)} \quad i = 1, 2, \dots, m-1$$

$$b_{i,j} = r\left(u_{i+1,j-1}^{(k-1)} + u_{i-1,j-1}^{(k)} - 2u_{i,j-1}^{(k-1)}\right) - 2u_{i,j-1}^{(k-1)} + 4u_{i,j}^{(k-1)} + 2k^2 f_{i,j}^{(k-1)}$$

b) SOR

$$u_{i,j+1}^{(k)} = \omega \cdot \overline{u}_{i,j} + (1-\omega)\, u_{i,j+1}^{(k-1)} \qquad i = \overline{1, m-1}$$

with $\quad \omega_{opt} = \dfrac{2}{1+\sqrt{1-t^2}}, \quad t = \dfrac{r}{1+r}\cos\left(\dfrac{\pi}{m}\right)$