Numerical Solution of Integral Equations.

Inverse Theory

- relatively recent topic
- many different kinds of equations + many different algorithms for the same problem

Classification:

1. Fredholm equations:

- definite integrals with fixed upper and lower limit

1.1. Fredholm equation of first kind (inhomogeneous):

$$\boxed{\underbrace{g(t)}_{\text{known}} = \int^{b} \underbrace{K(t,s)}_{\text{kernel}} f(s)\,ds} \quad (1)$$

(integral eq. have the right-hand side and the left-hand side are exchanged)

$f(t)$ - the unknown; continuous kernel function

(1) analogous to $K \cdot \vec{f} = \vec{g}$ with sol. $f = K^{-1} g$

(1) has a unique solution whenever $g \neq 0$ (for $g = 0$ (1) is almost always unusable) and K is invertible;

however in most cases K is not invertible

if $K(t,s) = K(t-s)$ and $\begin{matrix} a = -\infty \\ b = \infty \end{matrix} \Rightarrow f(t) = \int_{-\infty} \frac{F_t[g(t)](\omega)}{F_t[K(t)](\omega)} e^{2\pi i \omega t} d\omega$

(F_t - Fourier transform operator)

1.2. Fredholm equation of second kind

$$\boxed{f(t) = \lambda \int_a^b K(t,s) f(s)\,ds + g(t)} \quad (2)$$

analogous to $(K - \lambda \mathbb{1}) \cdot \vec{f} = \vec{g}$ (finite dim eigenvalue problem)

for $g = 0 \Rightarrow$ homogeneous equation

The solution of a general Fredholm int. eq. is called integral equation Neumann series

- 1.2.a) homogeneous $g=0$

if $k(t,s)$ bounded (mărginit), then eq. (2)

has solution for at most a denumerably set of infinite set λ_n $n=1,2,3,\ldots$ - eigenvalues

$\Rightarrow$ $f_n(t)$ eigenfunctions

if in addition $k(t,s)$ is symmetric $\Rightarrow$ λ_n are real

- 1.2 b) inhomogeneous $g(t) \neq 0$

– have solutions except for the case when λ is eigenvalue

$\Rightarrow$ the integral op / matrix is singular

(the Fredholm alternative)

eq. (1) are often ill-conditioned

$k \cdot f$ acts like a smoothing op $\Rightarrow$ loss of information

$\Rightarrow$ can't get it back in an inverse operation $\Rightarrow$

inverse problems

+ one uses extra information on the solution

"priori knowledge" used to restore lost inf

eq. (2) – hom. – likewise not ill-posed

– inhom. – not often —"—

2. Volterra equations

- special case of Fredholm eq. with $K(t,s) = 0$ for $s > t$
  $\Rightarrow$ upper limit can be rewritten as t

2.1. Volterra equation of first kind

$$\boxed{g(t) = \int_a^t K(t,s) f(s)\, ds} \quad (3)$$

analogous to $K \cdot \vec{f} = \vec{g}$ for components $\sum_{j=1}^{k} K_{kj} f_j = g_k$

- in fact Volterra eq corresponds to a matrix K that is lower triangular
  $\Downarrow$
  easy to solve with forward substitution

- eq. do not tend to be ill-conditioned

2.2. Volterra equation of second kind

$$\boxed{f(t) = \int_a^t K(t,s) f(s)\, ds + g(t)} \quad (4)$$

analogous to $\underbrace{(K - \mathbb{1})}_{\text{lower triangular}} \vec{f} = \vec{g}$

- λ is absorbed in K for inhomog. eq.
- if $g = 0$, a theorem states that eq (4) has no eigenval. with square-integrable eigenfunctions

$k(t,s)f(s)$ → linear integral equations

$k(t,s,f(s))$
$k(t,s,f(s),f(t))$ } non linear — " —
more complicated
not often in practice

- most methods to solve integrals numerically are based on quadratures, frequently Gaussian

- e.g. for solving the Fredholm equation of second kind with Simpson's rule:

- $$f(x) = \lambda \int_a^b k(x,y) f(y)\, dy + g(x) \quad (5), \quad x \in [a,b]$$

Simpson's rule (composite)

- $$\int_a^b f(x)\,dx = \frac{h}{3}\left(f_0 + f_{2n} + 4\sum_{i=1}^{n} f_{2i-1} + 2\sum_{i=1}^{n-1} f_{2i} + O(h^4)\right)$$

where $h = \frac{b-a}{2n}$ $\quad x_i = a + ih$ $\quad i = 0,1,\dots,2n$
$f_i = f(x_i)$
(6)

- define: $x_i = a + ih$ $\quad i,j = 0,1,\dots,2n$
$h = \frac{b-a}{2n}$ $\quad y_j = a + jh$

notation: $f(x_i) = f_i$ $\quad k(x_i, y_j) = k_{i,j}$
$g(x_i) = g_i$

(6) in (5)

$\Rightarrow$

$$\boxed{g_i = f_i - \frac{\lambda h}{3}\left(K_{i,0} f_0 + K_{i,2n} f_{2n} + 4\sum_{j=1}^{n} K_{i,2j-1} f_{2j-1} + 2\sum_{j=1}^{n-1} K_{i,2j} f_{2j}\right)}$$

$i = 0, 1, 2, \dots, 2n$ (7)

One has to solve the linear set of equations with unknowns $f_0, f_1, \dots, f_{2n}$ to determine approximations of $f(x)$ in the chosen points.

One can apply different quadrature rules . . .

- for Volterra equation the method is similar just that the upper limit of the integral is x_i $\Rightarrow i = 2n$ and in (7) one replaces n by $\frac{i}{2}$ . . and so on.