INTRODUCERE ÎN TEORIA CUANTICĂ A CÂMPURILOR ȘI A PARTICULELOR ELEMENTARE

INTRODUCTION TO QUANTUM FIELD THEORY AND ELEMENATRY PARTICLES

Anul universitar 2020-2021
Semestrul I

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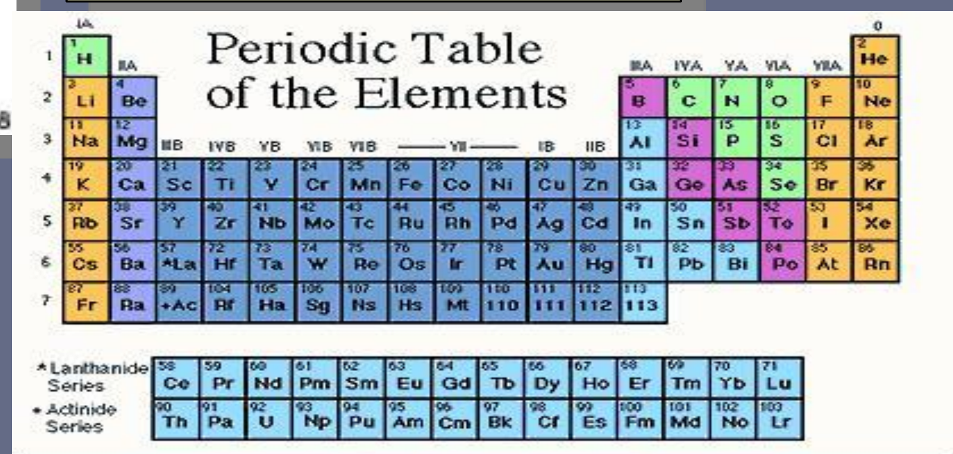
Introducere

- Din ce este constituită materia?
- Din ce este compus universul?
- Care este originea universului și cum a evoluat?
- De ce se comportă așa universul?
- Cum va evolua?
- ....

Care sunt elementele din care este constituita materia?

Empedocles 492-432 BC

> By convention there is color,
> by convention sweetness,
> by convention bitterness,
> but in reality there are
> atoms and space.
>
> Democritus 400 BC

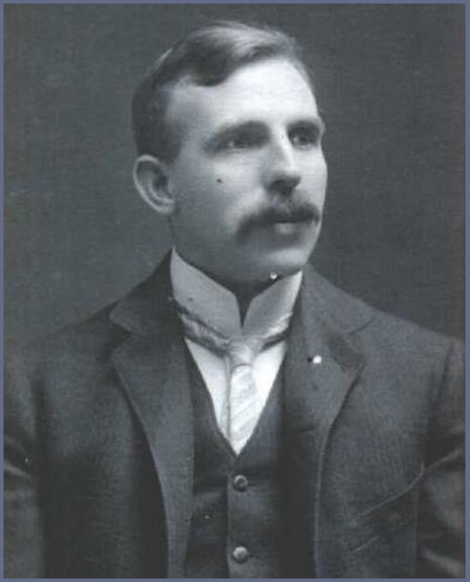


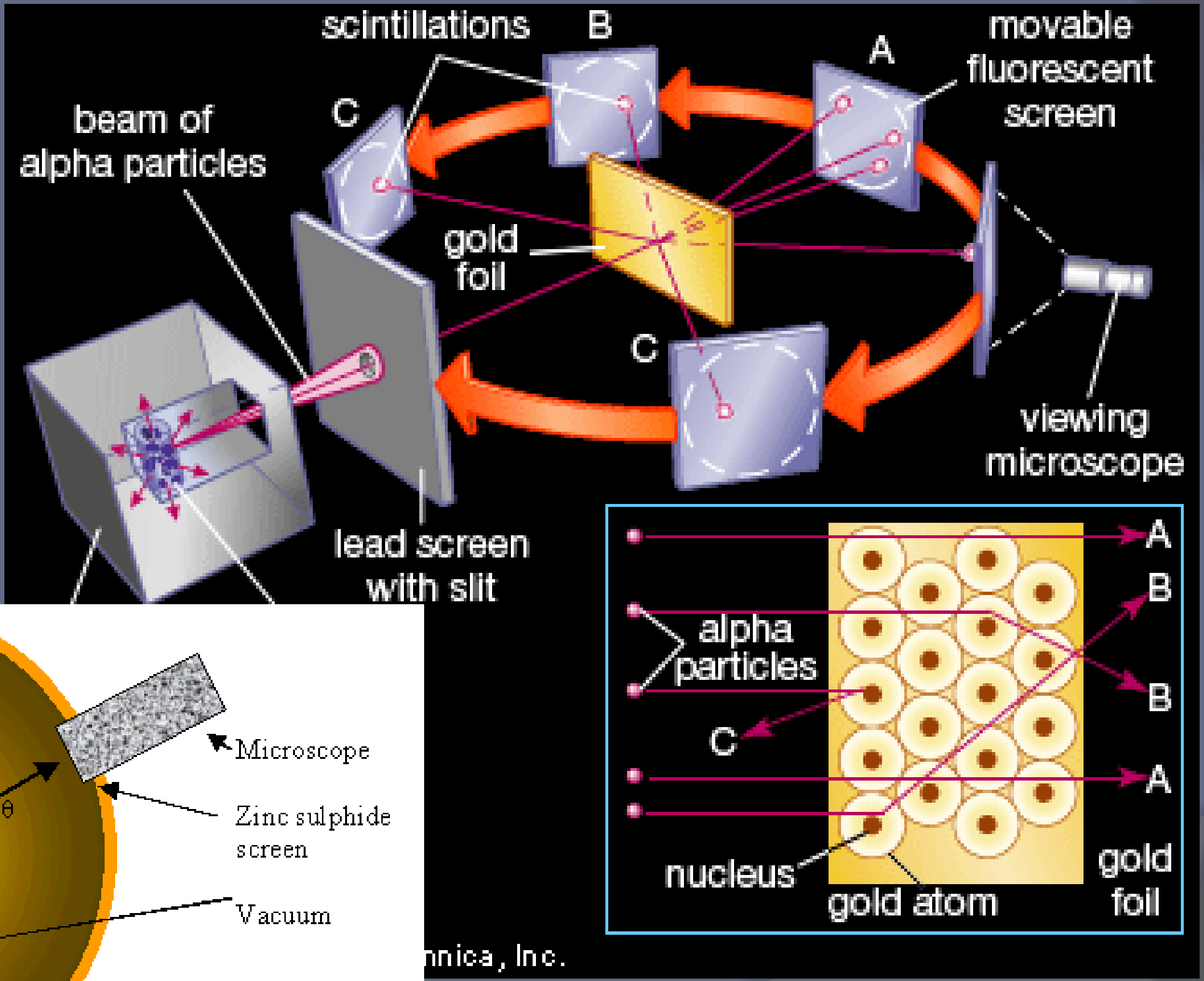
Mendeleev, 1869

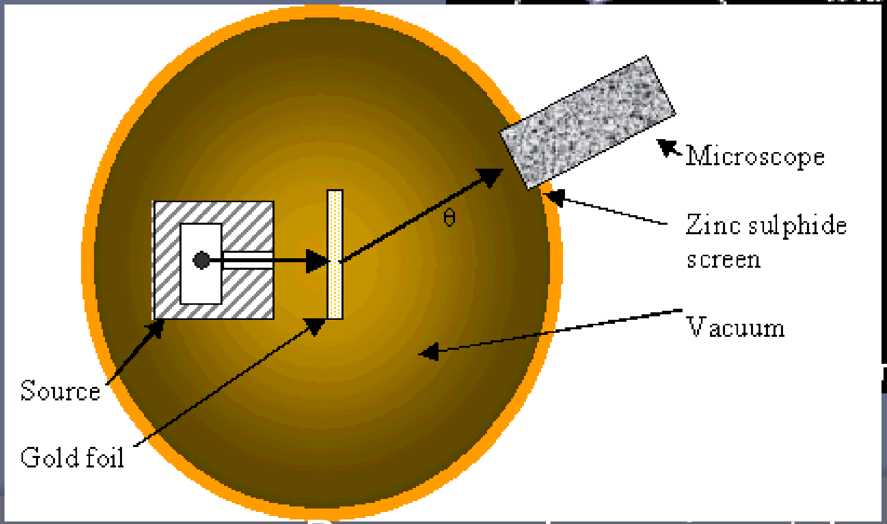
Scurt istoric

- **sfârșitul secolului XIX:**
  - **mecanică clasică;**
  - **electromagnetism;**
  - **termodinamică.**

1911 Rutherford: atomii nu sunt particule elementare!





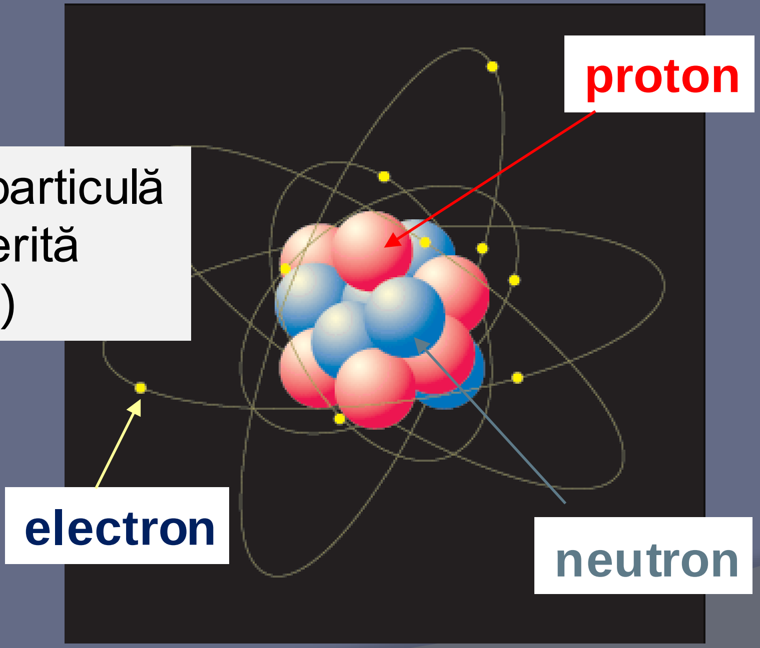


Precursorul experimentelor moderne de împrăștiere.

Atomii

Atomii:
- protoni și neutroni în nucleu
- electroni

Electronul - prima particulă elementară descoperită (J.J. Thomson 1897)



Sunt protonii și neutronii particule elementare?

Provocări

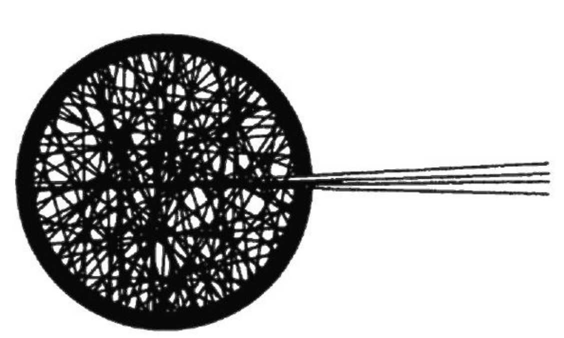
RADIATIA CORPULUI NEGRU

PROBLEMA ABSORBTIEI RADIATIEI

PROBLEMA EMISIEI RADIATIEI SI STABILITATII SISTEMELOR ATOMICE

Primele experimente

- **Radiația corpului negru (1895, 1900)**



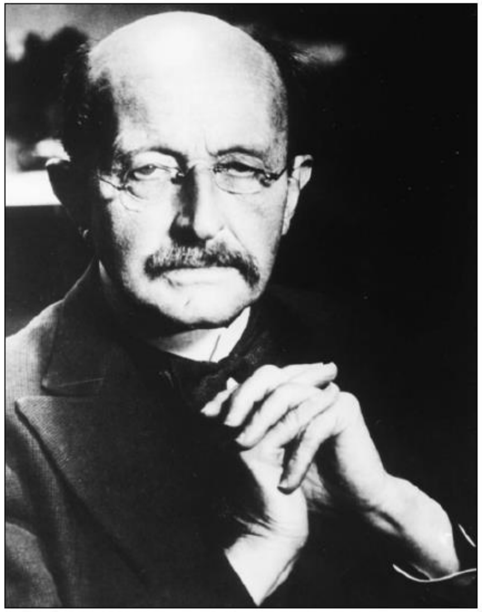
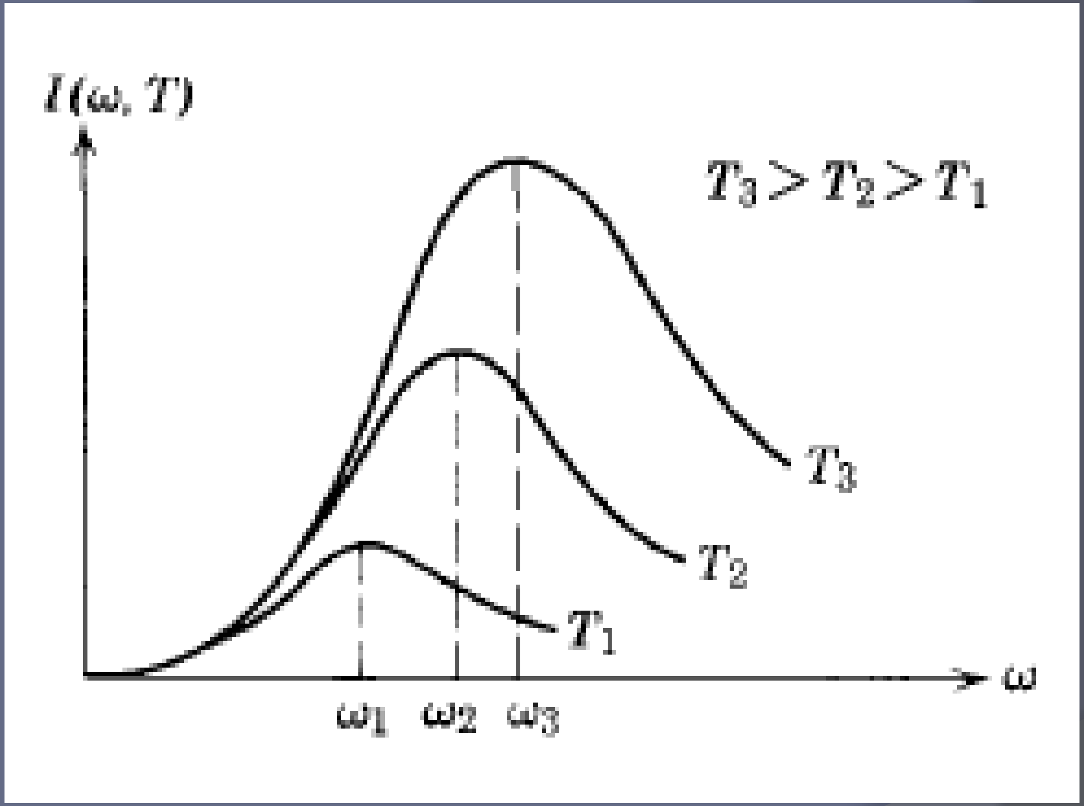


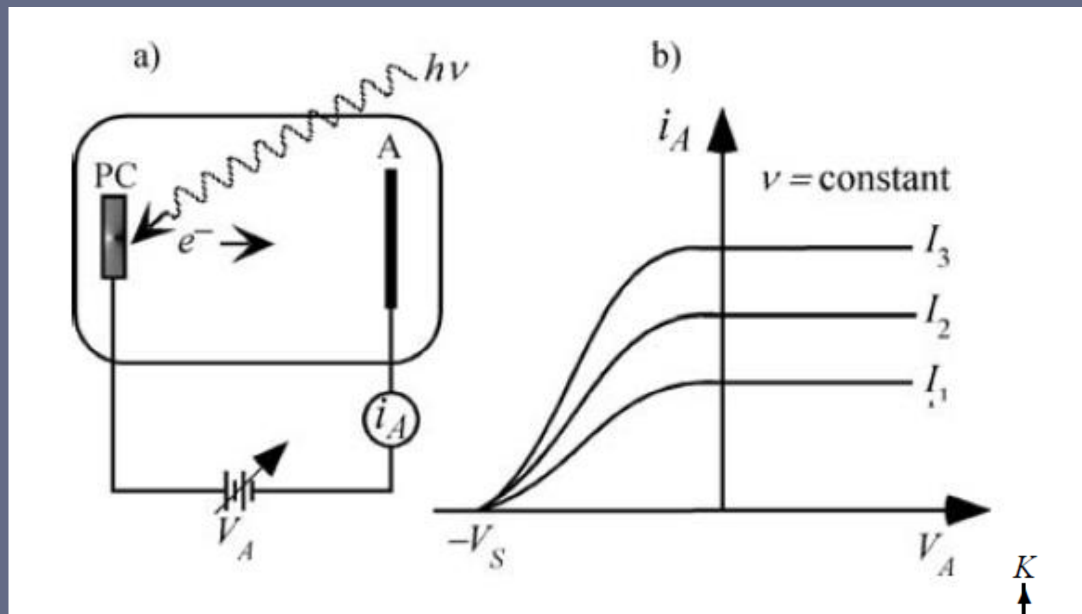
Figure 1.1: Max Planck. AIP Emilio Segre Visual Archives.

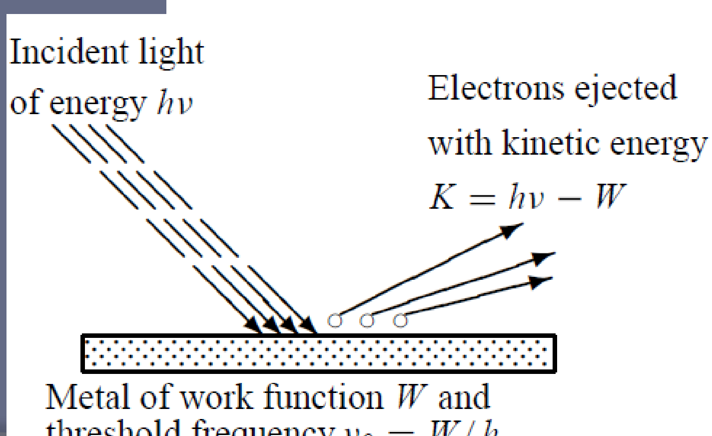


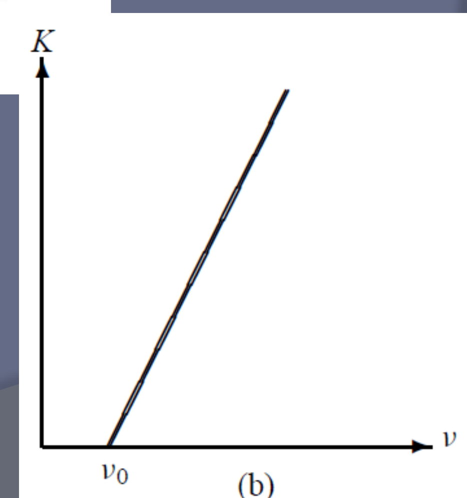
$$\frac{du(\nu,T)}{d\nu}=\frac{8\pi\nu^2}{c^3}h\nu\frac{1}{e^{\frac{h\nu}{kT}}-1}$$

Primele experimente

- **Efectul fotoelectric (1887,1902,1905)**

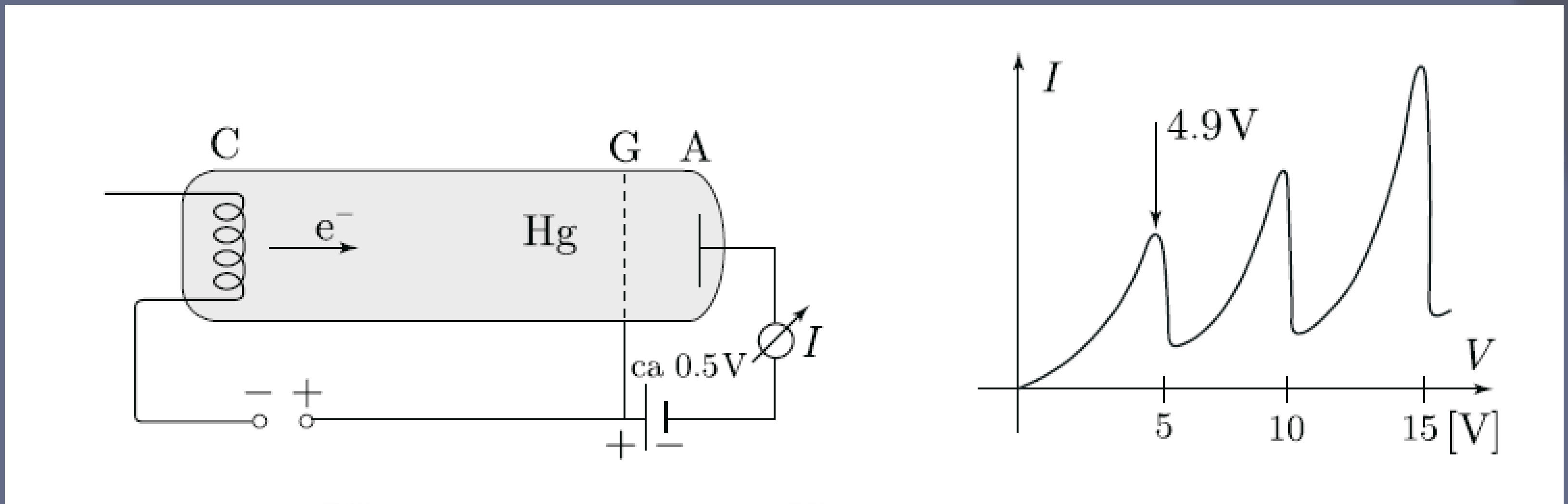


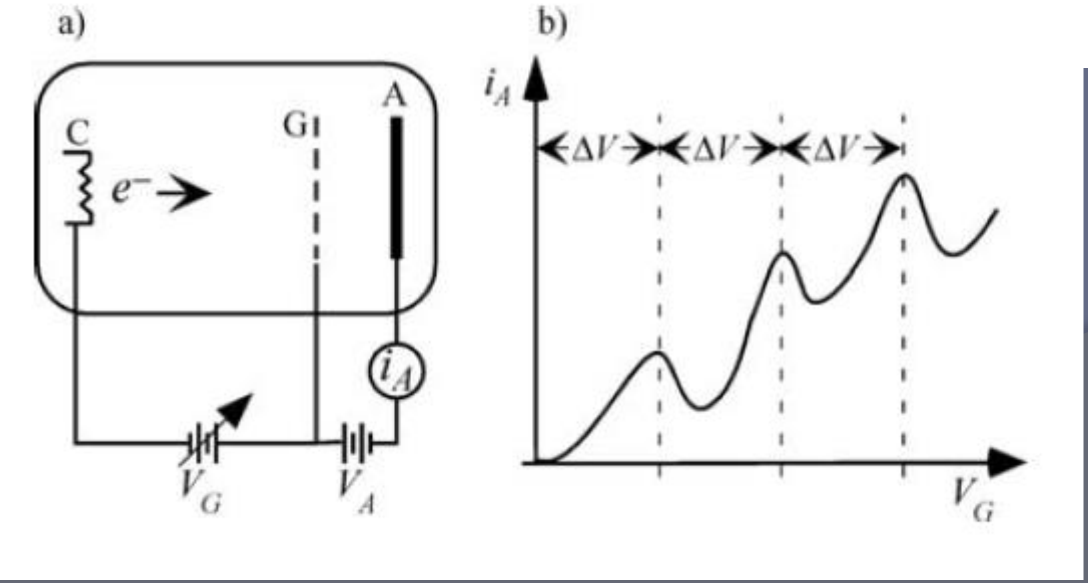




Primele experimente

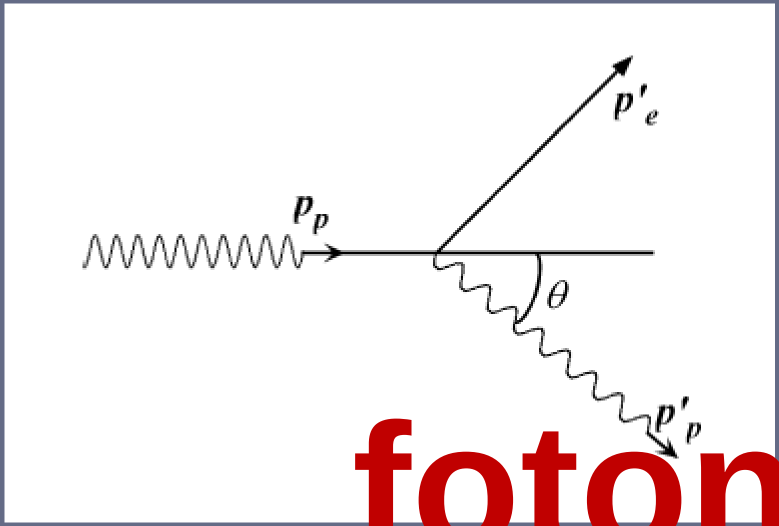
- **Experimentul FRANK-HERTZ (1914)**

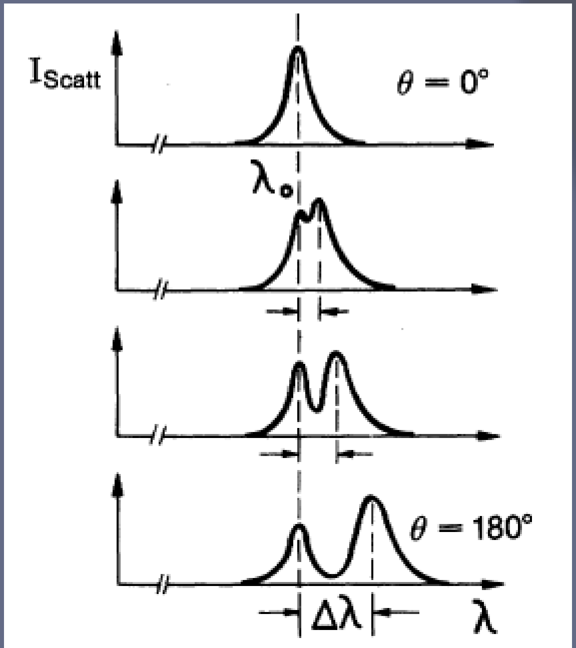




Primele experimente

- **Efectul Compton (1922)**





fotonul

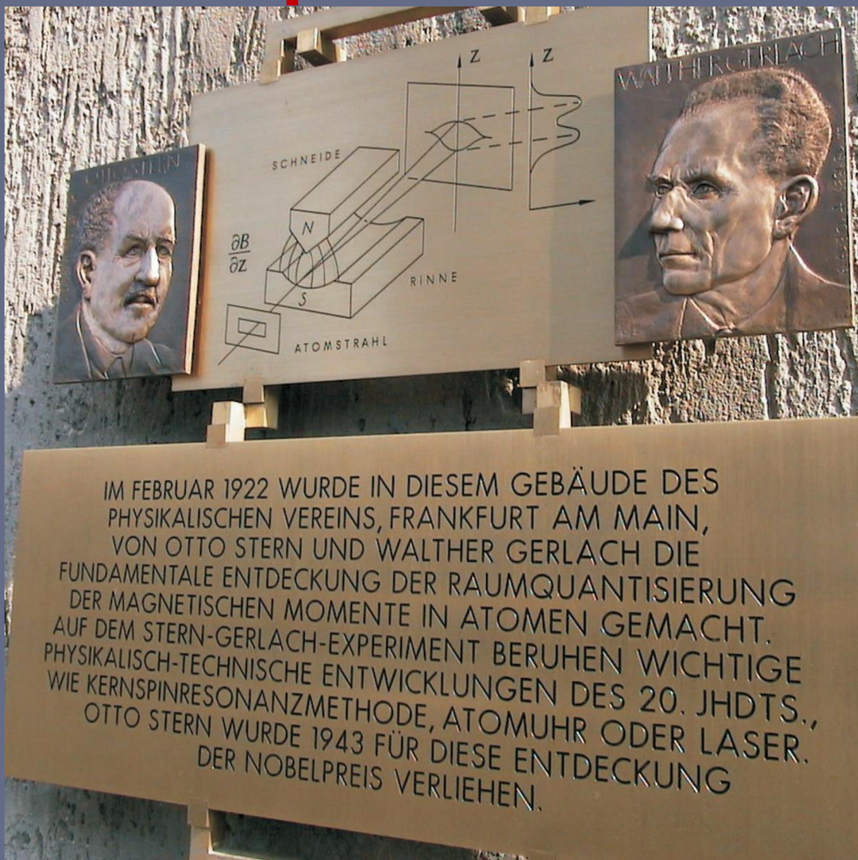
1926 – G.Lewis

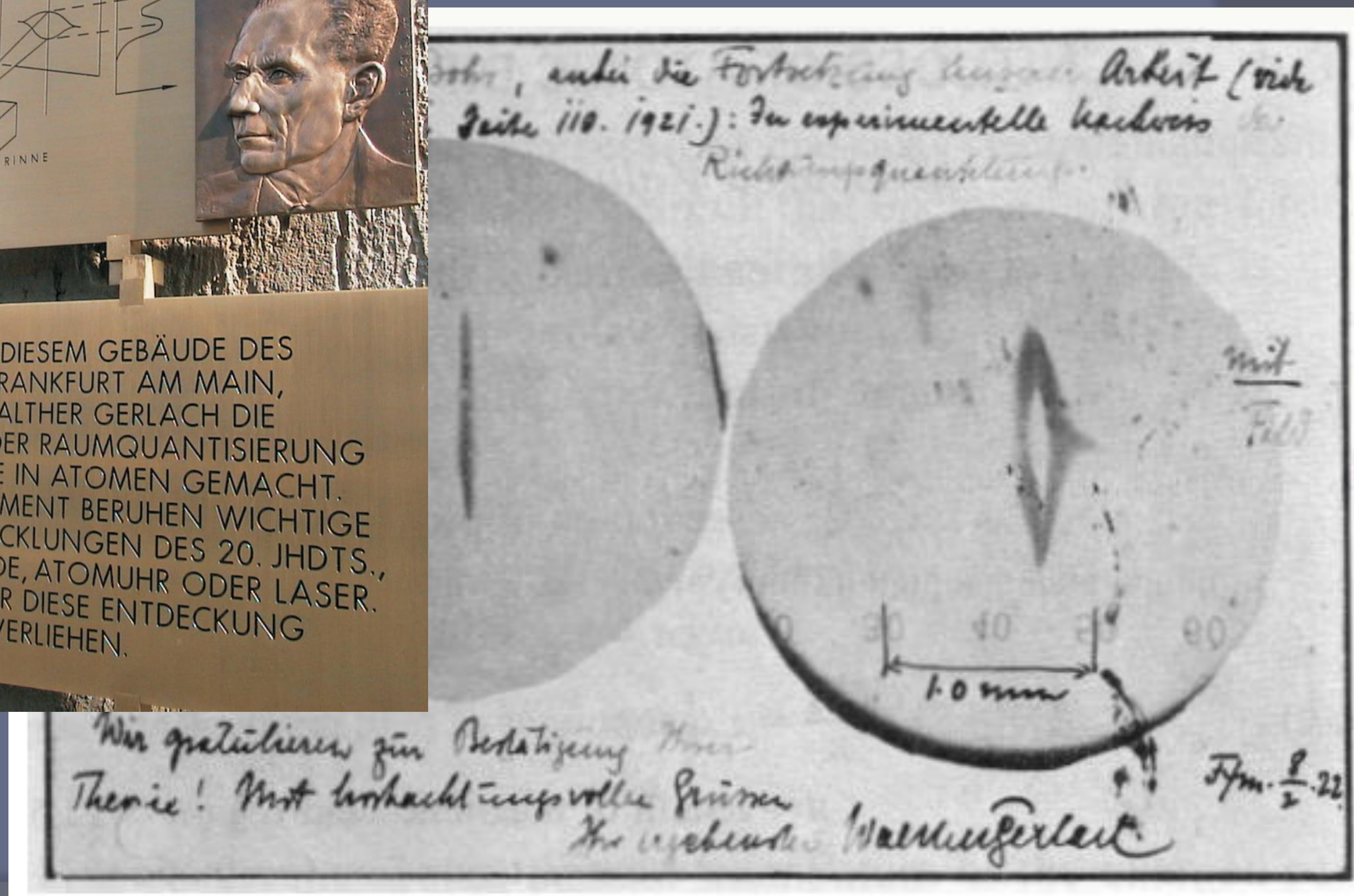
$$\lambda_c = \frac{h}{m_e c}$$

$$\Delta\lambda = \lambda_c(1 - \cos\theta)$$

Primele experimente

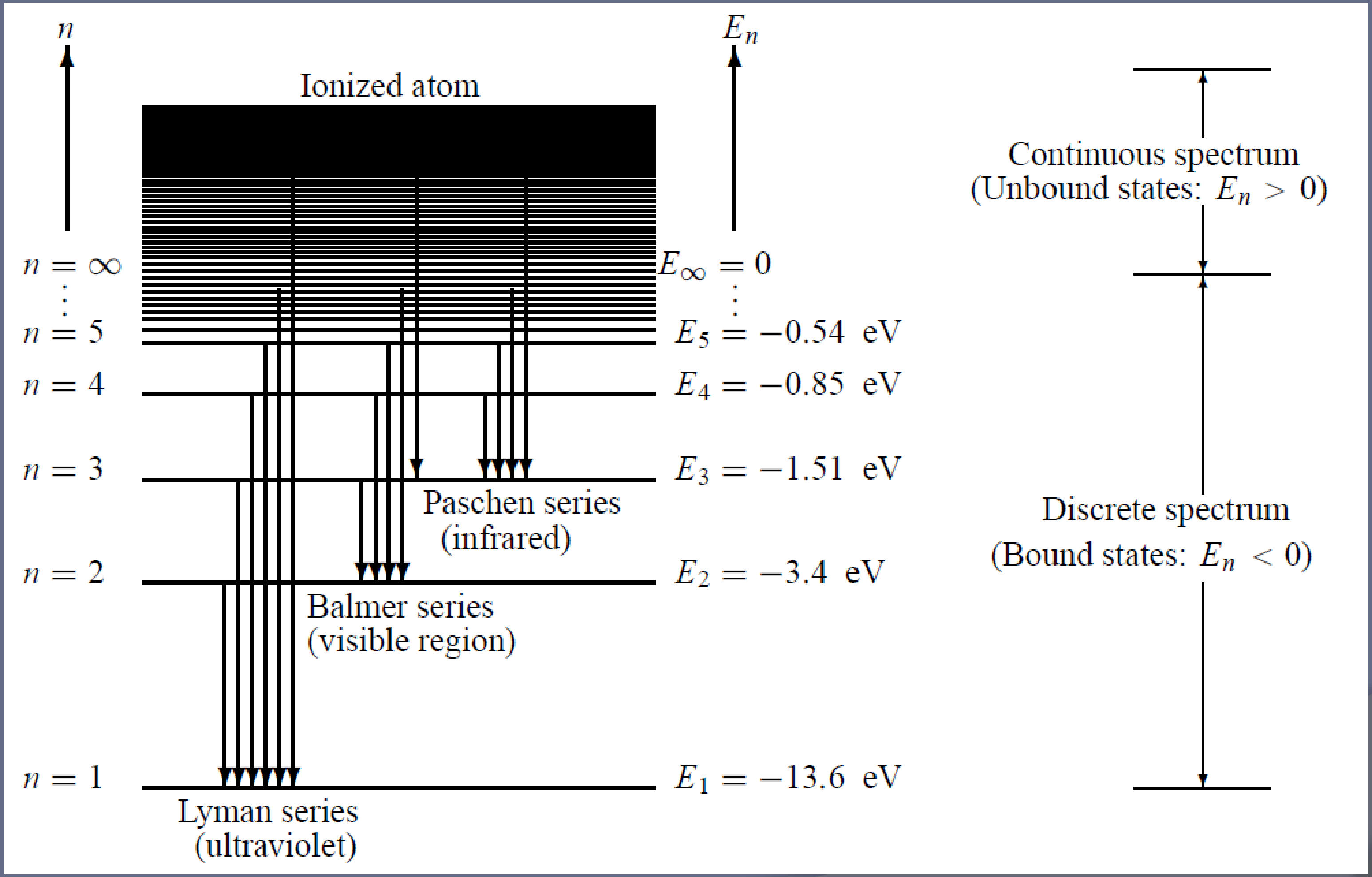
- **Experimentul STERN - GERLACH (1922)**





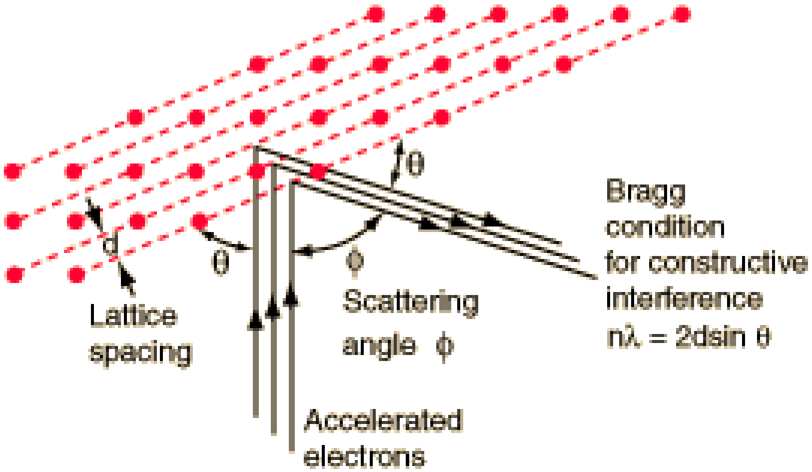
Primele experimente

- **Spectroscopie atomică**

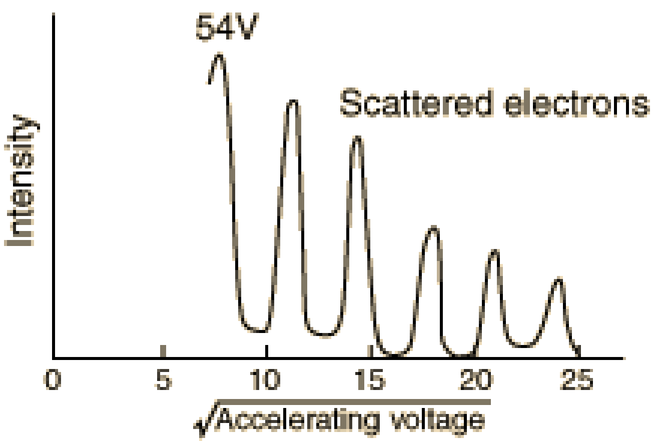


Primele experimente

- **Experimentele de difracție (1927)**



Davisson and Germer
Phys.Rev. (1927)



$$\frac{1}{\lambda} = \frac{n}{2d\sin\theta} = \frac{p}{h} = \frac{\sqrt{2mE}}{h} = \frac{\sqrt{2meV}}{h}$$

Electron wavelength | *Bragg law* | *deBroglie relationship* | *Acceleration through voltage V*

Davisson, C. J., "Are Electrons Waves?," Franklin Institute Journal 205, 597 (1928)

Scurt istoric

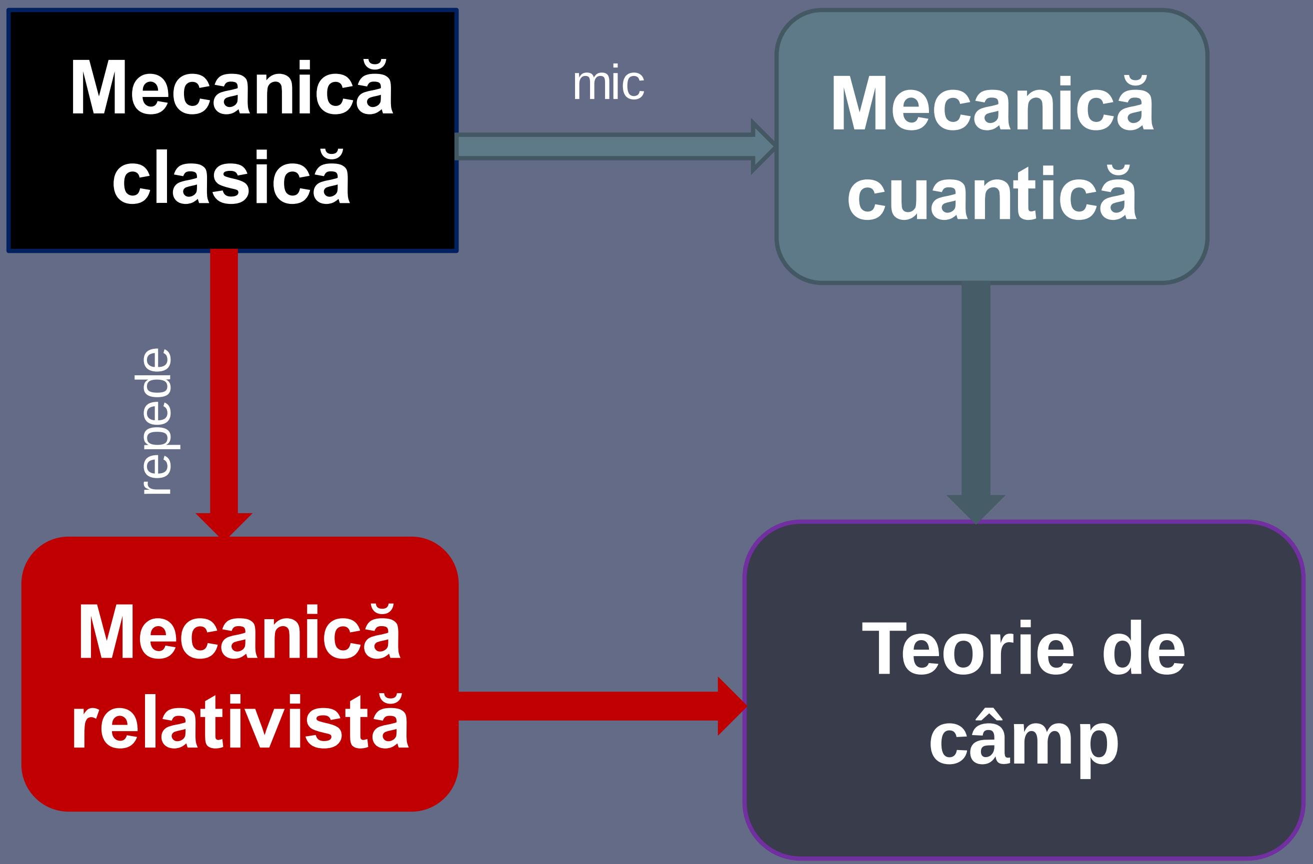
- **sfârșitul secolului XIX:**
  - **mecanică clasică;**
  - **electromagnetism;**
  - **termodinamică.**
- **începutul secolului XX:**
  - **domeniul relativist** **(mecanica Newtoniană nu poate fi folosită la viteze foarte mari)**
  - **domeniul microscopic** **(fizica clasică nu poate fi folosită la nivel microscopic – e.g. pentru descrierea atomilor și moleculelor, a interacției cu câmpul electromagnetic etc.)**

Ce legi folosim?

Ce mecanică folosim?

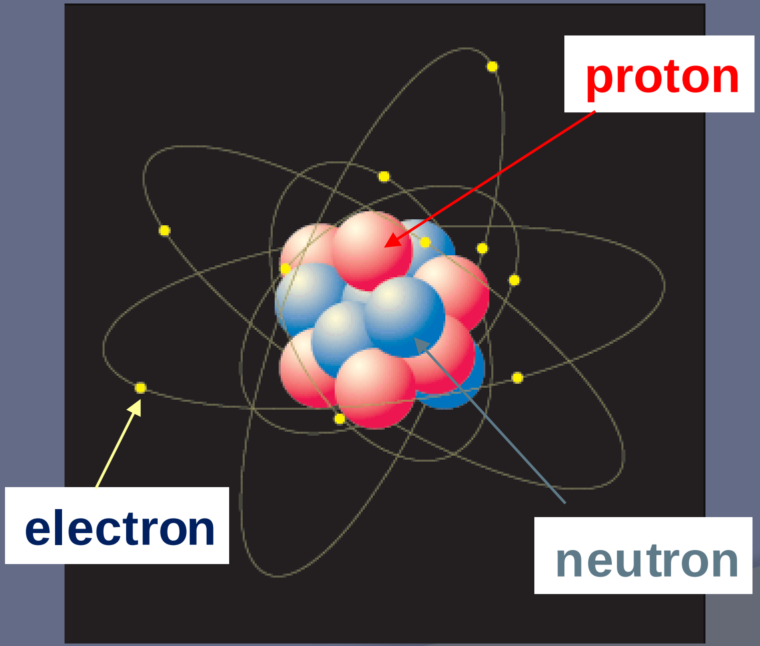
Legea atracției universale

Legile de mișcare ale lui Newton – mecanica clasică


Mecanică clasică
mic
Mecanică cuantică
repede
Mecanică relativistă
Teorie de câmp

Atomii

Atomii:
- protoni și neutroni în nucleu
- electroni



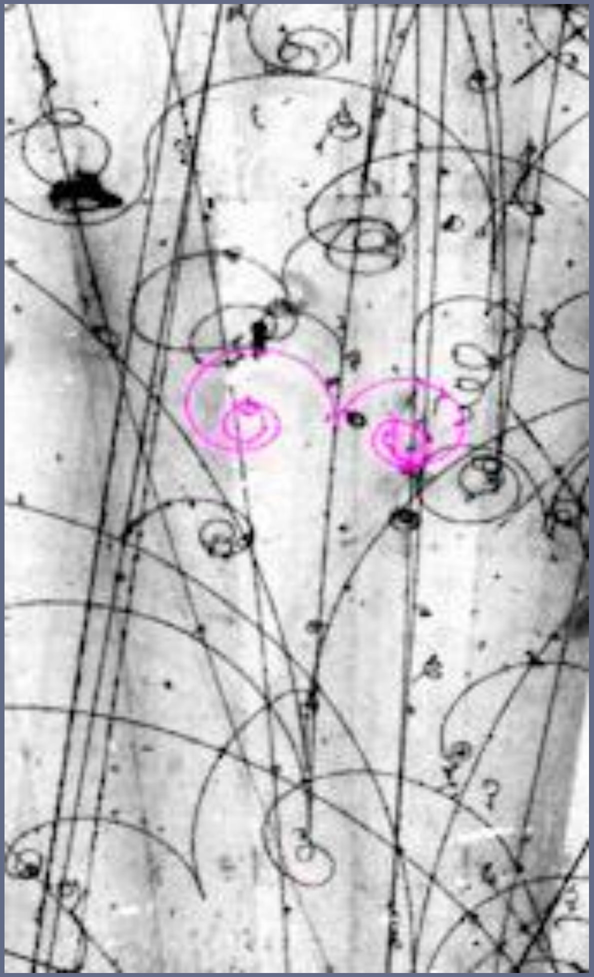
Sunt protonii și neutronii particule elementare?

Fizica particulelor elementare

1. Care sunt particulele elementare (ce proprietăți au – masă, sarcină electrică, spin, ...)?
2. Cum interacționează?
3. Cum producem particule elementare?
4. Cum detectăm particule elementare?

Dirac – particulă – antiparticulă

sarcină electrică de semn opus



• Pereche electron-positron creată din fotoni într-o cameră cu bule.

• Energia fotonului este transformată în materie și anti-materie.

• Energia și impulsul se conservă (dar nu și masa de repaus)

Yukawa – 1934

- Ce ține protonii și neutronii în nucleul?

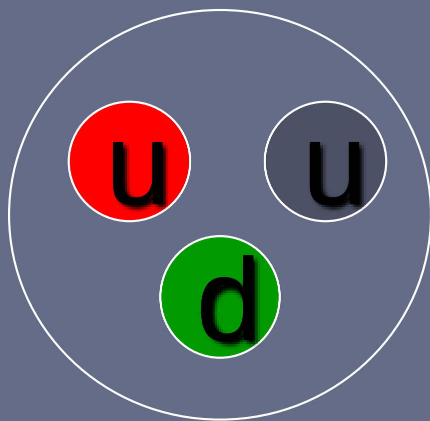
FORȚA TARE

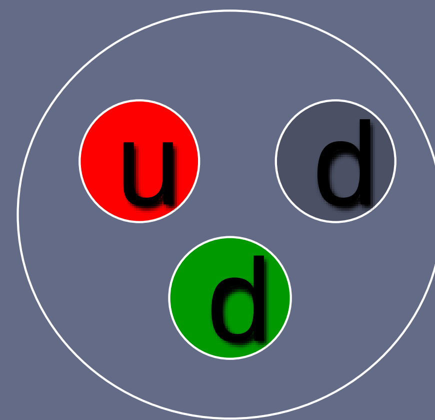
- De ce nu o experimentăm în viața de zi cu zi?

acționează la distanțe scurte

Protonii și neutronii – modelul cuarcilor (1964)

proton (sarcină +1) **neutron (sarcină 0)**





Cuarcii au sarcini electrice fracționare

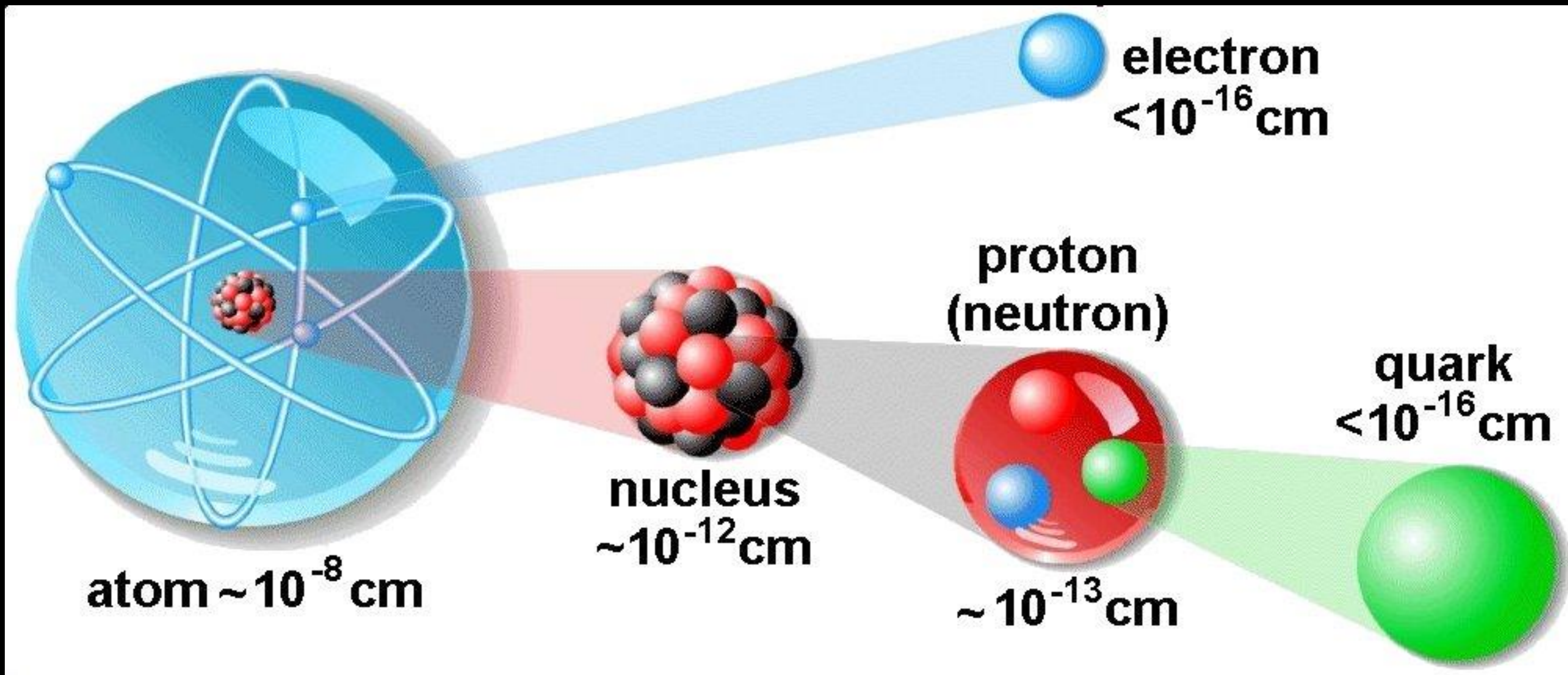
u - cuarcul up – sarcină electrică + 2/3

d - cuarcul down – sarcină electrică –1/3

$$u\left(+\frac{2}{3}\right)u\left(+\frac{2}{3}\right)d\left(-\frac{1}{3}\right)=p(+1)$$

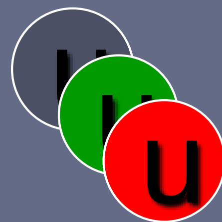
$$u\left(+\frac{2}{3}\right)d\left(-\frac{1}{3}\right)d\left(-\frac{1}{3}\right)=n(0)$$

Structura materiei (astăzi!)



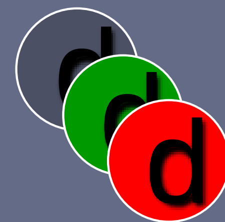
Cuarcii și culorile

Fiecare cuarc poate avea 3 “culori”



up

+2/3



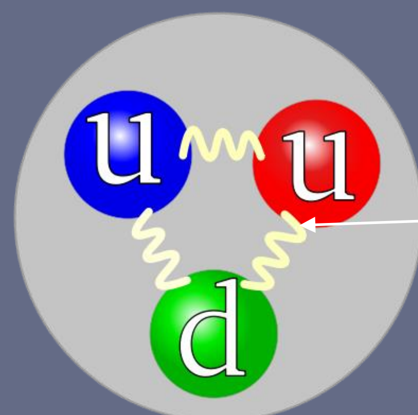
down

-1/3

Cuarcii se combină în așa fel încât să formeze particule „incolore” (confinare).

-***Barioni*** (3 cuarci qqq: roșu + verde + albastru = alb)

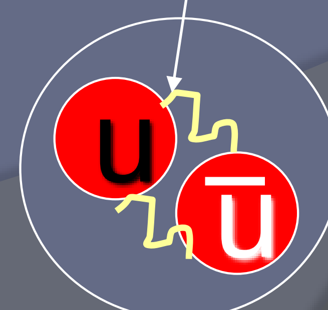
proton
p



Forța tare “lipește” cuarcii în stări legate

-***Mezoni*** (cuarc-anticuarc qqbar)

pion
π



PDG – Particle Data Group

STRANGE MESONS

K_L^0	130
K_S^0	310
K^0	311
K^+	321
$\mathbf{K_0^*(800)^0}$	**9000311***
$\mathbf{K_0^*(800)^+}$	**9000321***
$K_0^*(1430)^0$	10311
$K_0^*(1430)^+$	10321
$K(1460)^0$	100311
$K(1460)^+$	100321
$\mathbf{K(1830)^0}$	**9010311***
$\mathbf{K(1830)^+}$	**9010321***
$K_0^*(1950)^0$	**9020311***
$K_0^*(1950)^+$	**9020321***
$K^*(892)^0$	313
$K^*(892)^+$	323
$K_1(1270)^0$	10313
$K_1(1270)^+$	10323
$K_1(1400)^0$	20313
$K_1(1400)^+$	20323
$K^*(1410)^0$	100313
$K^*(1410)^+$	100323
$\mathbf{K_1(1650)^0}$	**9000313***
$\mathbf{K_1(1650)^+}$	**9000323***
$K^*(1680)^0$	30313

CHARMED MESONS

D^+	411
D^0	421
$\mathbf{D_0^*(2400)^+}$	10411
$\mathbf{D_0^*(2400)^0}$	10421
$D^*(2010)^+$	413
$D^*(2007)^0$	423
$D_1(2420)^+$	10413
$D_1(2420)^0$	10423
$D_1(H)^+$	20413
$\mathbf{D_1(2430)^0}$	20423
$D_2^*(2460)^+$	415
$D_2^*(2460)^0$	425
D_s^+	431
$D_{s0}^*(2317)^+$	10431
D_s^{*+}	433
$D_{s1}(2536)^+$	10433
$D_{s1}(2460)^+$	20433
$D_{s2}^*(2573)^+$	435

BOTTOM MESONS

B^0	511
B^+	521
B_0^{*0}	10511
B_0^{*+}	10521
B^{*0}	513

$c\bar{c}$ MESONS

$\eta_c(1S)$	441
$\chi_{c0}(1P)$	10441
$\eta_c(2S)$	100441
$J/\psi(1S)$	443
$h_c(1P)$	10443
$\chi_{c1}(1P)$	20443
$\psi(2S)$	100443
$\psi(3770)$	30443
$\psi(4040)$	9000443
$\psi(4160)$	9010443
$\psi(4415)$	9020443
$\chi_{c2}(1P)$	445
$\mathbf{\chi_{c2}(2P)}$	**100445***

$b\bar{b}$ MESONS

$\eta_b(1S)$	551
$\chi_{b0}(1P)$	10551
$\eta_b(2S)$	100551
$\chi_{b0}(2P)$	110551
$\eta_b(3S)$	200551
$\chi_{b0}(3P)$	210551
$\Upsilon(1S)$	553
$h_b(1P)$	10553
$\chi_{b1}(1P)$	20553
$\Upsilon_1(1D)$	30553

LIGHT BARYONS

p	2212
n	2112
Δ^{++}	2224
Δ^+	2214
Δ^0	2114
Δ^-	1114

STRANGE BARYONS

Λ	3122
Σ^+	3222
Σ^0	3212
Σ^-	3112
Σ^{*+}	3224^d
Σ^{*0}	3214^d
Σ^{*-}	3114^d
Ξ^0	3322
Ξ^-	3312
Ξ^{*0}	3324^d
Ξ^{*-}	3314^d
Ω^-	3334

CHARMED BARYONS

Λ_c^+	4122
Σ_c^{++}	4222
Σ_c^+	4212
Σ_c^0	4112
Σ_c^{*++}	4224
Σ_c^{*+}	4214

BOTTOM BARYONS

Λ_b^0	5122
Σ_b^-	5112
Σ_b^0	5212
Σ_b^+	5222
Σ_b^{*-}	5114
Σ_b^{*0}	5214
Σ_b^{*+}	5224
Ξ_b^-	5132
Ξ_b^0	5232
$\Xi_b^{\prime -}$	5312
$\Xi_b^{\prime 0}$	5322
Ξ_b^{*-}	5314
Ξ_b^{*0}	5324
Ω_b^-	5332
Ω_b^{*-}	5334
Ξ_{bc}^0	5142
Ξ_{bc}^+	5242
$\Xi_{bc}^{\prime 0}$	5412
$\Xi_{bc}^{\prime +}$	5422
Ξ_{bc}^{*0}	5414
Ξ_{bc}^{*+}	5424
Ω_{bc}^0	5342
$\Omega_{bc}^{\prime 0}$	5432

Universul este alcătuit numai din cuarci și electroni?

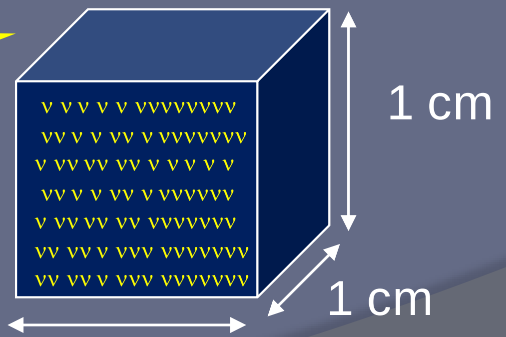
Există și neutrini!

ν

Electronul, protonul și neutronul sunt rari!
Pentru fiecare dintre ei, există 1 billion neutrini.

Neutrinii sunt cele mai abundente particule ale materiei în univers.

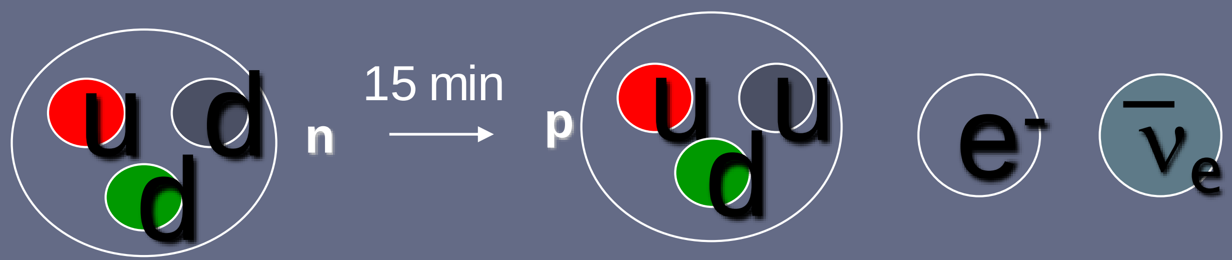
În fiecare cm^3 al spațiului sunt ~300 neutrini de la Big Bang



Neutrinii sunt peste tot.

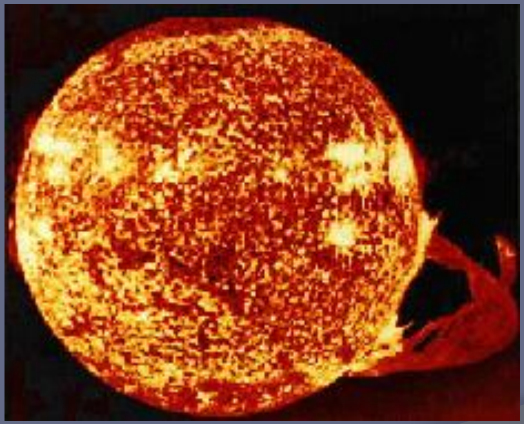
Dezintegrarea β

La nivelul cuarcilor: $d \to u\, e^- \, \bar{\nu}_e$



Un neutron se dezintegrează în 15 minute.

interacție „slabă”!



!

Tipuri de forțe

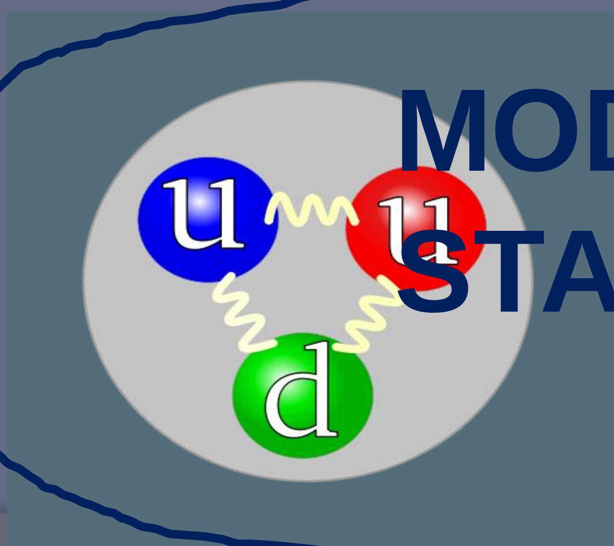
Gravitațională

masa

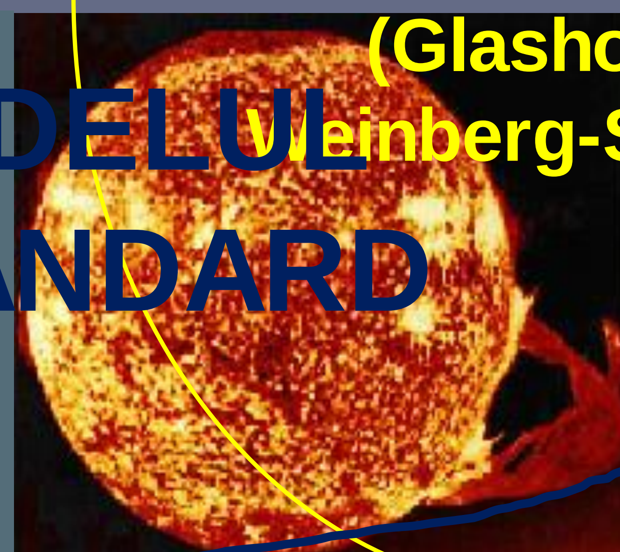
Electromagnetică

ELECTROSLABĂ (Glashow-Weinberg-Salam)

Tare



MODELUL STANDARD



Slabă

Cine mediază interacțiile?

Gravitațională

? graviton ?

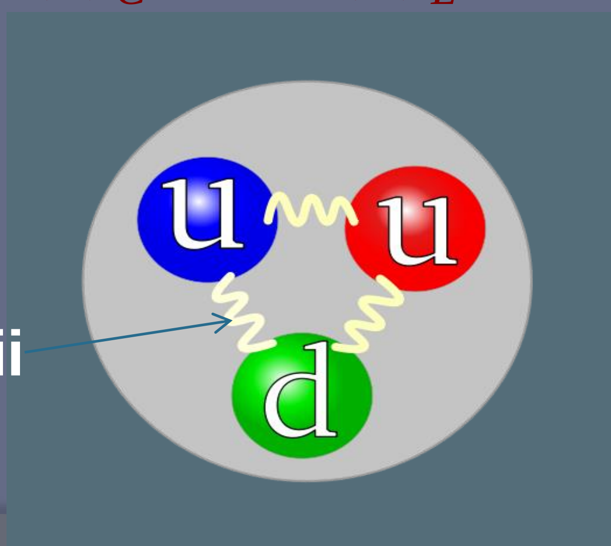
Electromagnetică

γ- fotonul

$$G_{SM} = SU(3)_C \times SU(2)_L \times U(1)$$

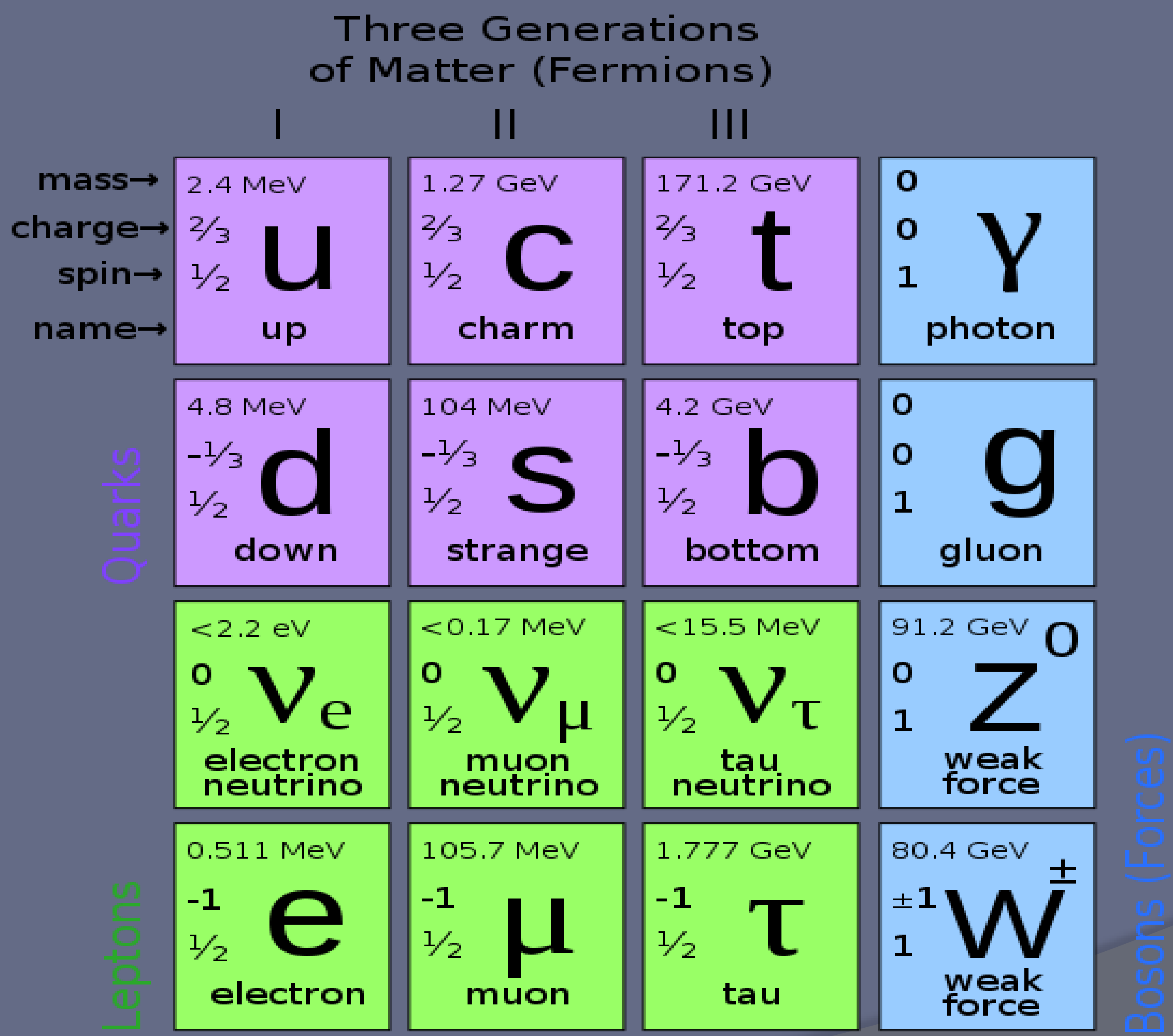
Tare

gluonii



Slabă

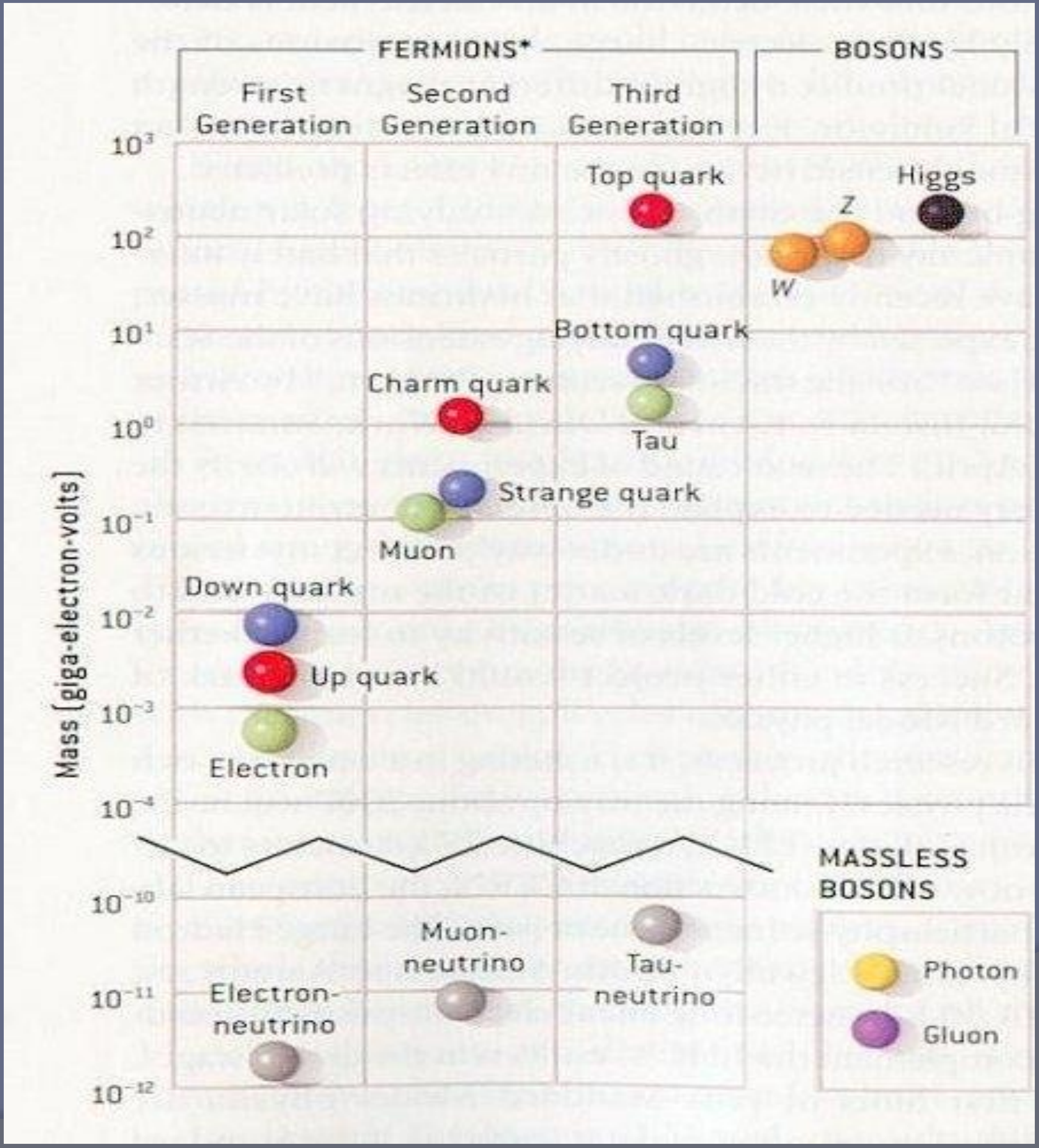
W^+, W^-, Z


Three Generations
of Matter (Fermions)
I
II
III
mass→
charge→
spin→
name→
2.4 MeV
⅔
½
u
up
1.27 GeV
⅔
½
c
charm
171.2 GeV
⅔
½
t
top
0
0
1
γ
photon
4.8 MeV
-⅓
½
d
down
104 MeV
-⅓
½
s
strange
4.2 GeV
-⅓
½
b
bottom
0
0
1
g
gluon
<2.2 eV
0
½
νe
electron neutrino
<0.17 MeV
0
½
νμ
muon neutrino
<15.5 MeV
0
½
ντ
tau neutrino
91.2 GeV
0
1
Z0
weak force
0.511 MeV
-1
½
e
electron
105.7 MeV
-1
½
μ
muon
1.777 GeV
-1
½
τ
tau
80.4 GeV
±1
1
W±
weak force
Quarks
Leptons
Bosons (Forces)

3 FAMILII (ASTĂZI!)

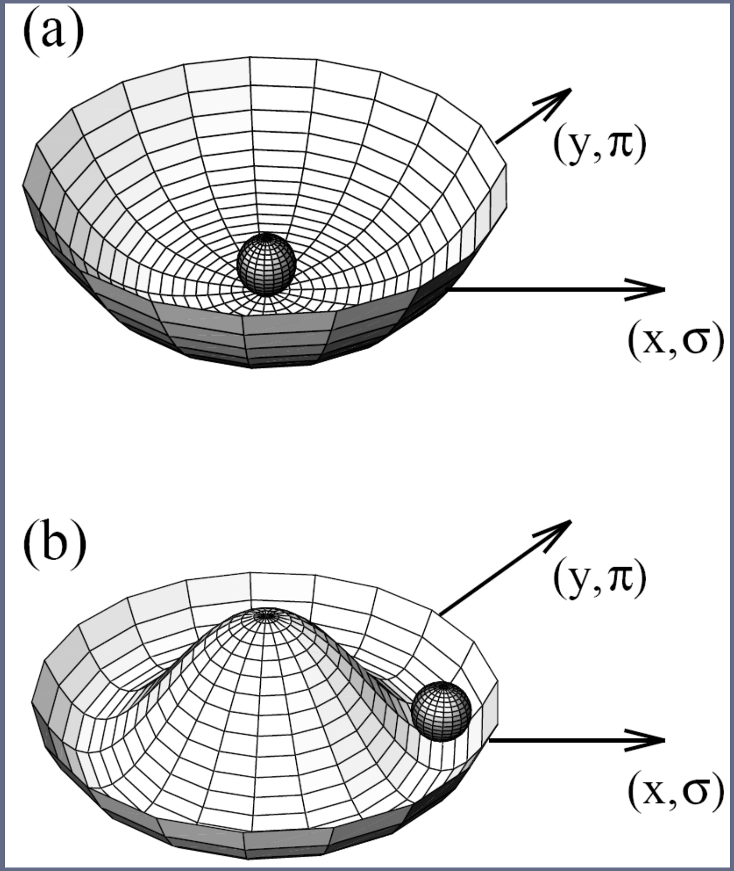
prima generație	a doua generație	a treia generație
0 ν_e	0 ν_μ	0 ν_τ
+2/3 u	+2/3 c	+2/3 t
-1 e^-	-1 μ^-	-1 τ^-
-1/3 d	-1/3 s	-1/3 b
materie „obișnuită”	radiație cosmică	acceleratori

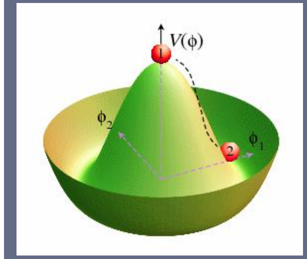
cele 3 generații diferă prin masă!



FERMIONS*
BOSONS
First Generation
Second Generation
Third Generation
Mass (giga-electron-volts)
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-10}
10^{-11}
10^{-12}
Top quark
Higgs
Z
W
Bottom quark
Charm quark
Tau
Strange quark
Muon
Down quark
Up quark
Electron
MASSLESS BOSONS
Muon-neutrino
Tau-neutrino
Electron-neutrino
Photon
Gluon

De unde apare masa particulelor în teorie?



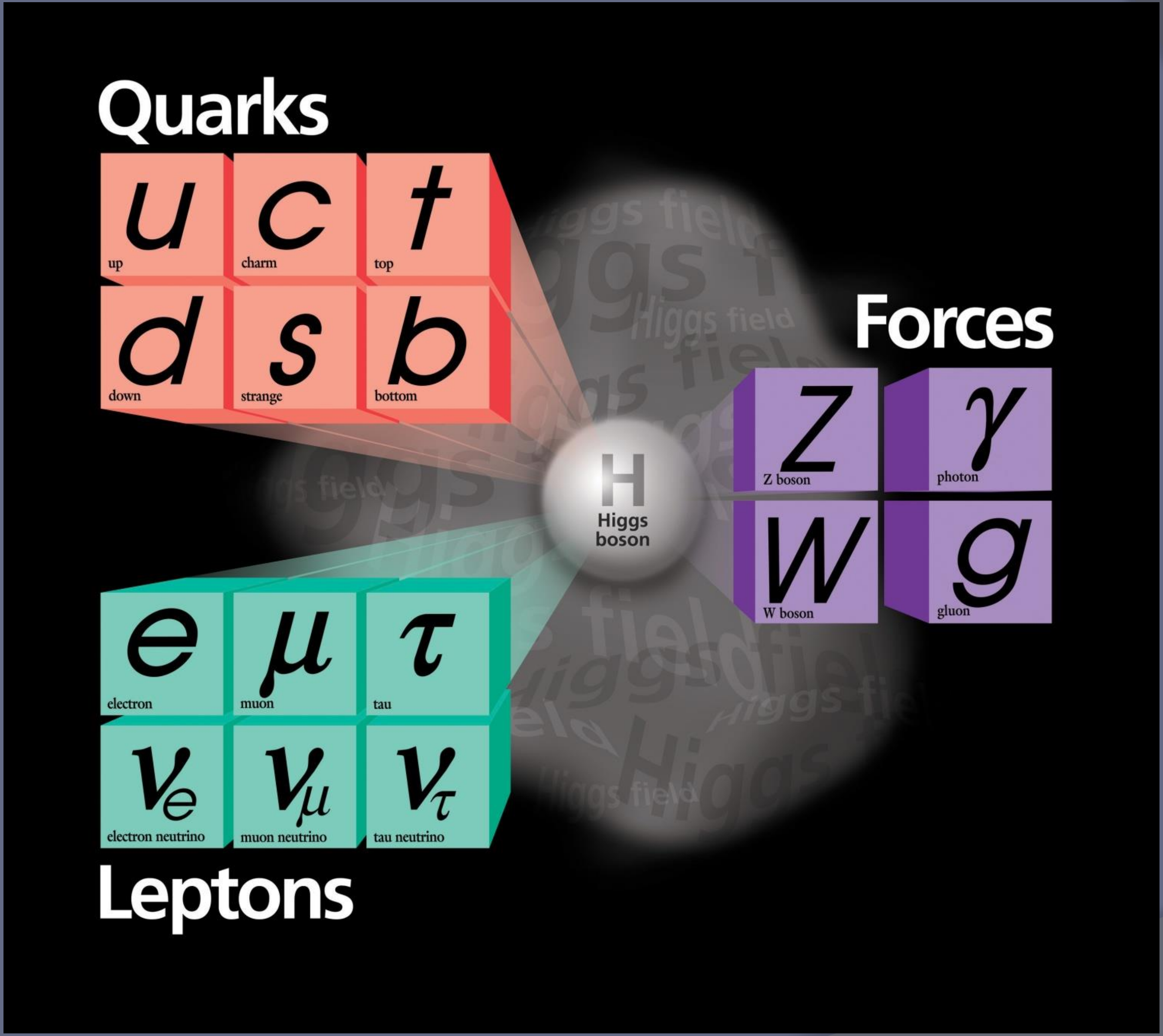


W^+, W^-, Z – au masă

- rezultă bozonul Higgs

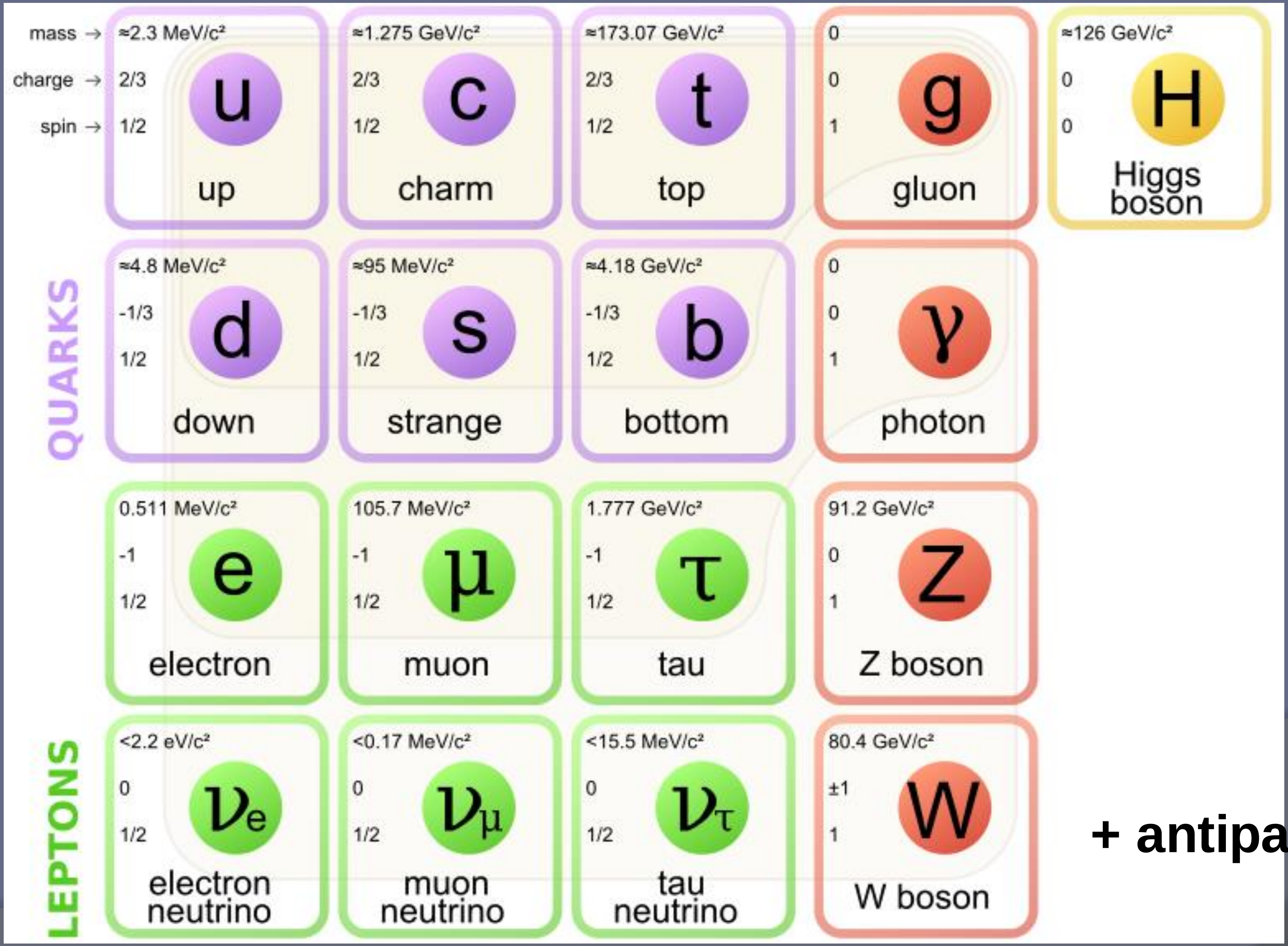
fără rupere spontană de simetrie

rupere spontană de simetrie



Quarks
u
up
c
charm
t
top
d
down
s
strange
b
bottom
Forces
Z
Z boson
γ
photon
W
W boson
g
gluon
H
Higgs boson
Higgs field
e
electron
μ
muon
τ
tau
νe
electron neutrino
νμ
muon neutrino
ντ
tau neutrino
Leptons

Modelul Standard



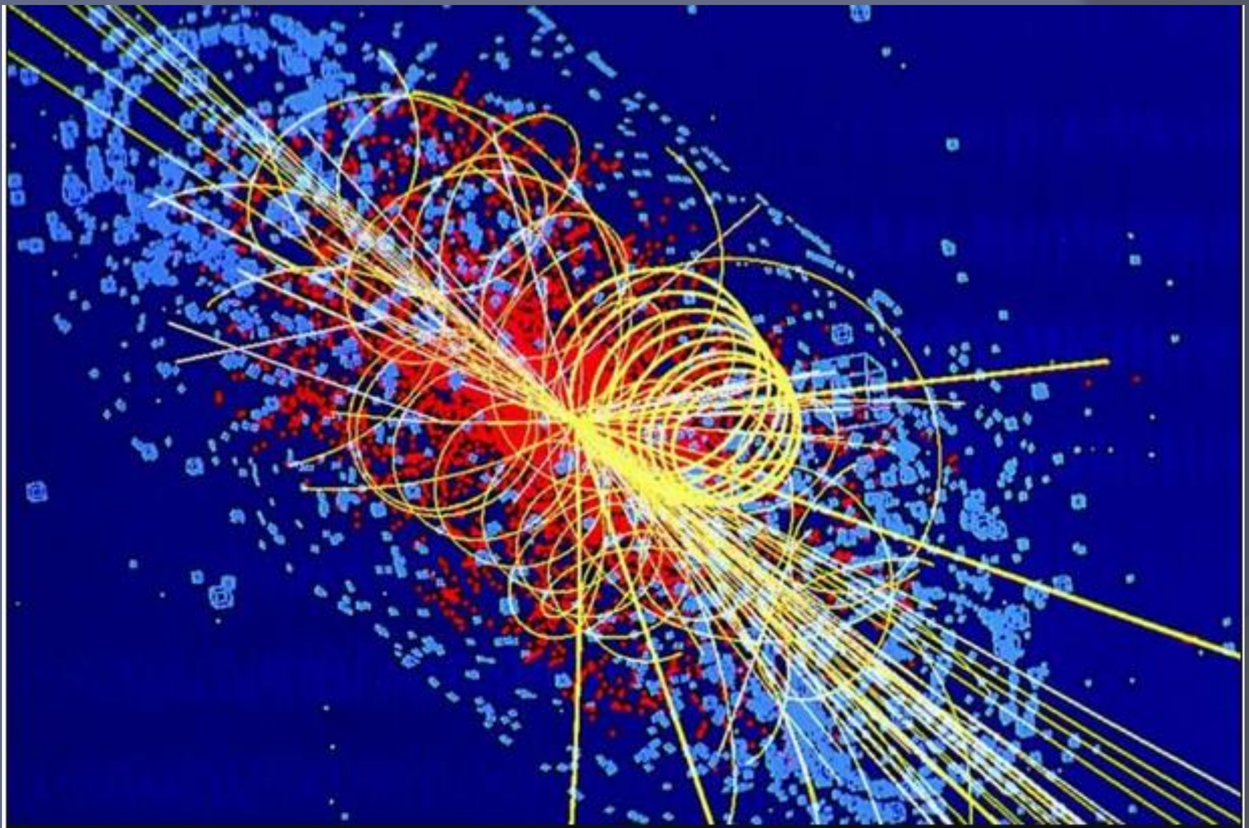
+ antiparticule!

Fizica particulelor elementare

1. Care sunt particulele elementare (ce proprietăți au – masă, sarcină electrică, spin, ...)?
2. Cum interacționează? - De unde obținem informații?
3. Cum producem particule elementare?
4. Cum detectăm particule elementare?

2. De unde obținem informații despre particule?

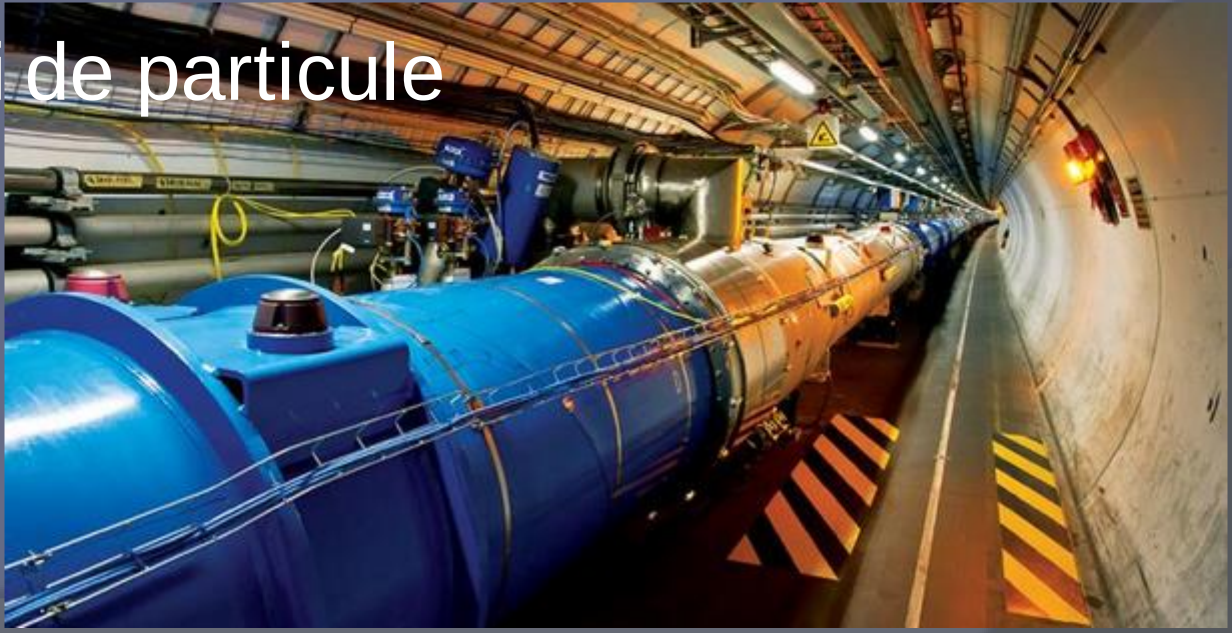
- ciocniri
- dezintegrări
- stări legate



Simulation of a particle collision in which a Higgs boson is produced (Image: Lucas Taylor/CMS)

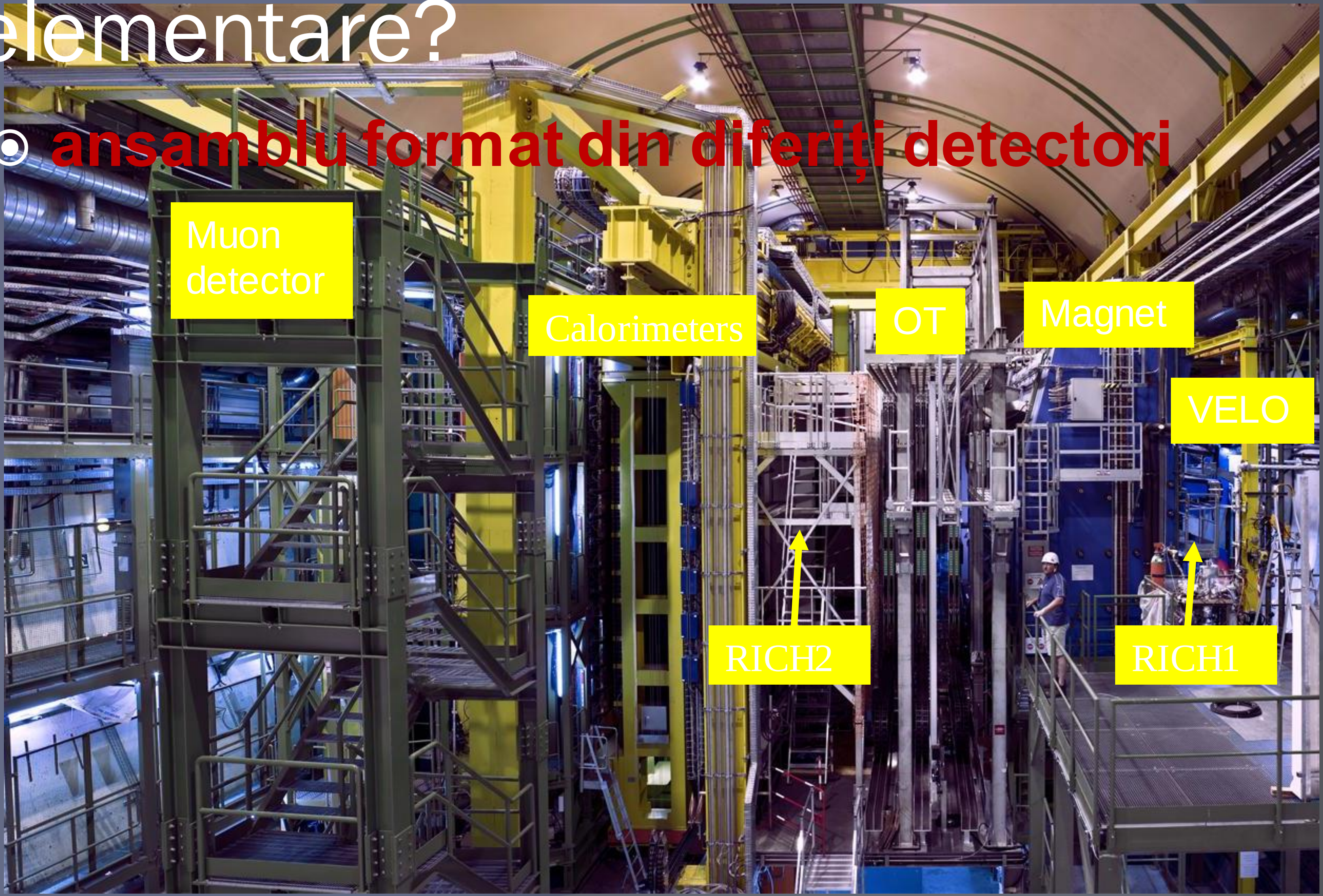
3. Cum producem particule elementare?

- metode simple pentru electroni, protoni (e.g. ionizări)
- radiații cosmice
- reactori nucleari
- acceleratori de particule



4. Cum detectăm particule elementare?

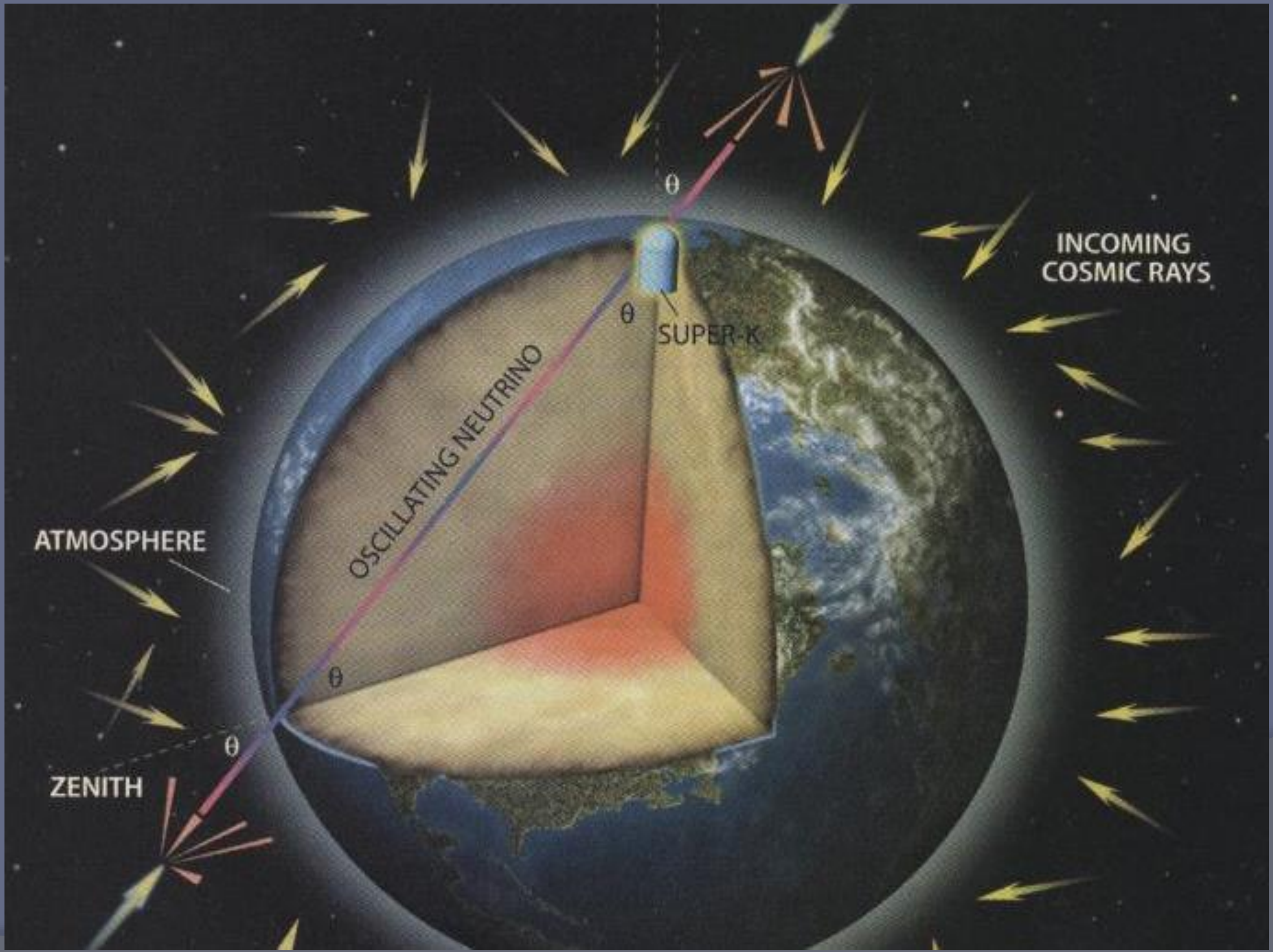
- **ansamblu format din diferiți detectori**



Modelul Standard – răspunsul la toate întrebările?

- **NU!**

Cum acomodăm în teorie masa neutrinilor?



De ce în univers există mai multă materie decât antimaterie?

- Există cantități mari de materie, dar nu și dovada unor cantități mari de antimaterie.

violarea conservării sarcinii și parității

CP – charge-parity

beauty

B^0

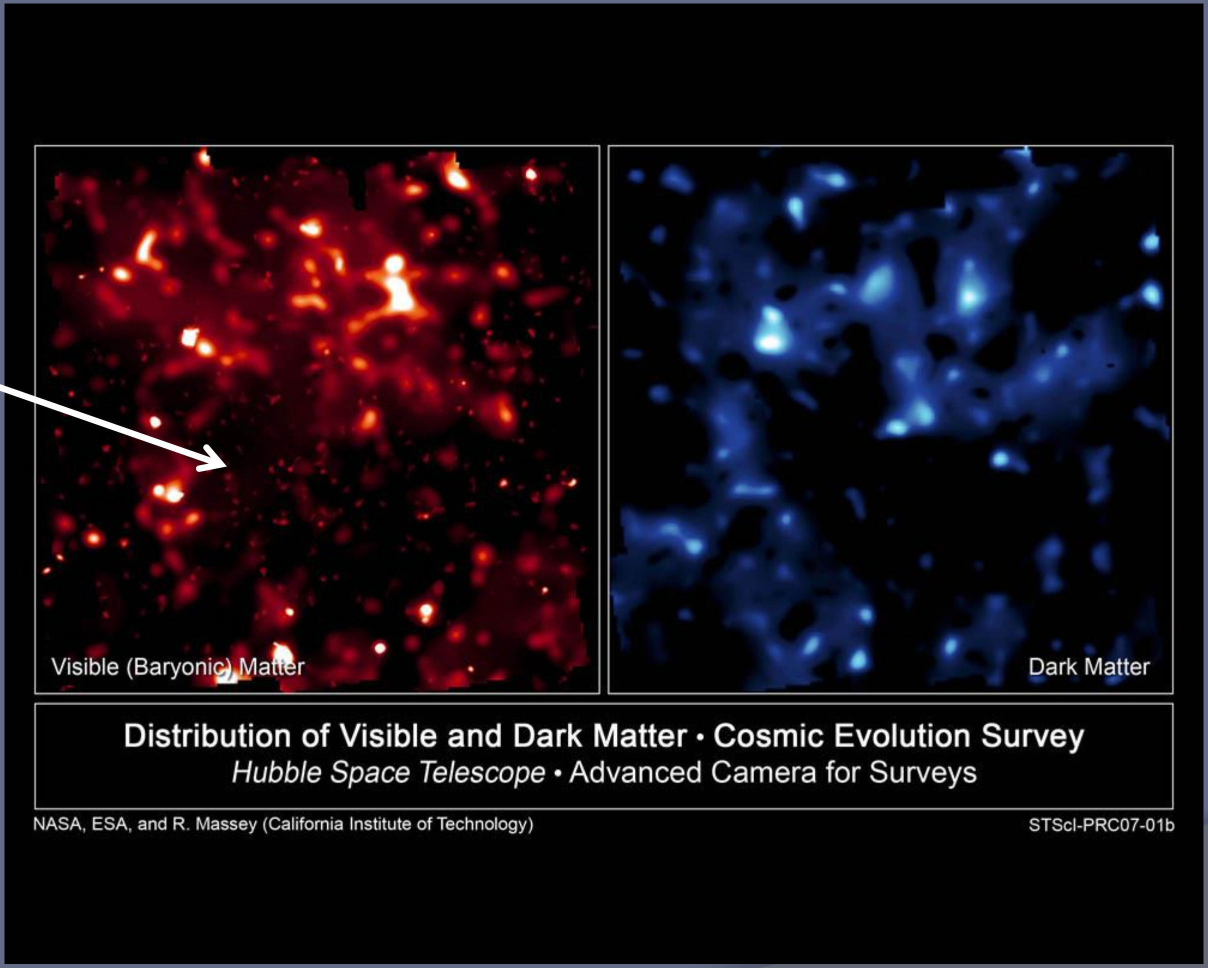
CP

anti-beauty

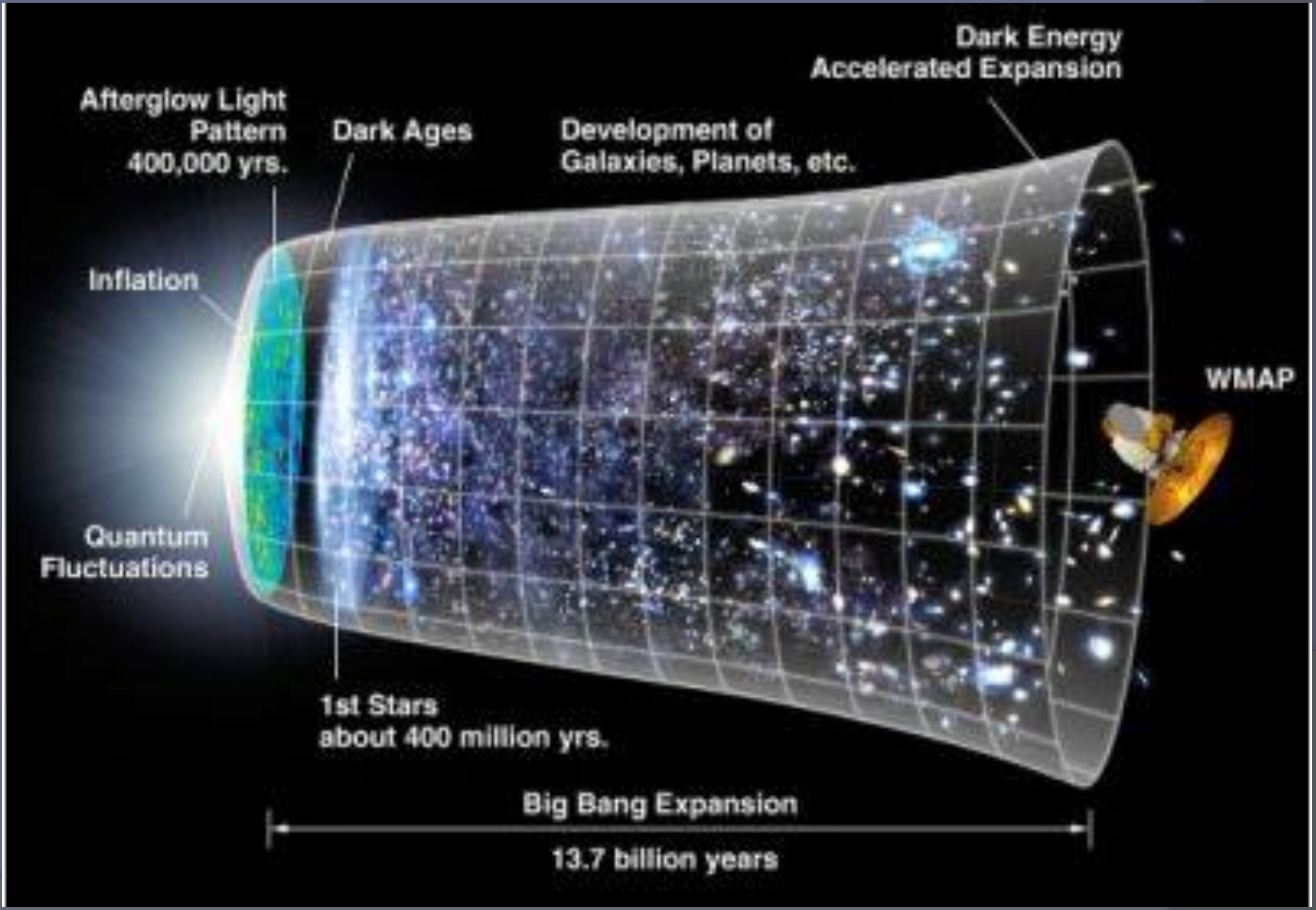
$\overline{B^0}$

Ce este „dark matter”?

4%



Ce face universul astăzi?



Introducere în teoria cuantică a câmpurilor și a particulelor elementare

- ***Simetrii spațiu-timp***

Grupul Lorentz (GL) și grupul Poincare (GP). Teorema Noether. Tensorul energie-impuls. Momentul cinetic. Simetrii interne.

- ***Câmpuri clasice libere***

Câmpul scalar real și complex, câmpul Weyl, câmpul Dirac, câmpul Proca, câmpul electromagnetic. Cuantificarea câmpurilor fundamentale.

- ***Introducere în teoriile de etalonare***

- Principiul invarianței la transformări de etalonare locale. Derivata covariantă. Interacțiile fundamentale în cazul grupurilor de etalonare U(1), SU(2) și SU(3).
- Ruperea spontană a unei simetrii globale. Teorema Goldstone.
- Teorii de etalonare cu rupere spontană a simetriei. Mecanismul Higgs.
- Bazele modelului standard al particulelor elementare și al interacțiilor dintre acestea.

$$\mathcal{L}_{GWS} = \sum_f (\bar{\Psi}_f (i\gamma^\mu \partial\mu - m_f)\Psi_f - eQ_f \bar{\Psi}_f \gamma^\mu \Psi_f A_\mu) +$$

$$+ \frac{g}{\sqrt{2}} \sum_i (\bar{a}_L^i \gamma^\mu b_L^i W_\mu^+ + \bar{b}_L^i \gamma^\mu a_L^i W_\mu^-) + \frac{g}{2c_w} \sum_f \bar{\Psi}_f \gamma^\mu (I_f^3 - 2s_w^2 Q_f - I_f^3 \gamma_5) \Psi_f Z_\mu +$$

$$-\frac{1}{4} |\partial_\mu A_\nu - \partial_\nu A_\mu - ie(W_\mu^- W_\nu^+ - W_\mu^+ W_\nu^-)|^2 - \frac{1}{2} |\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ +$$

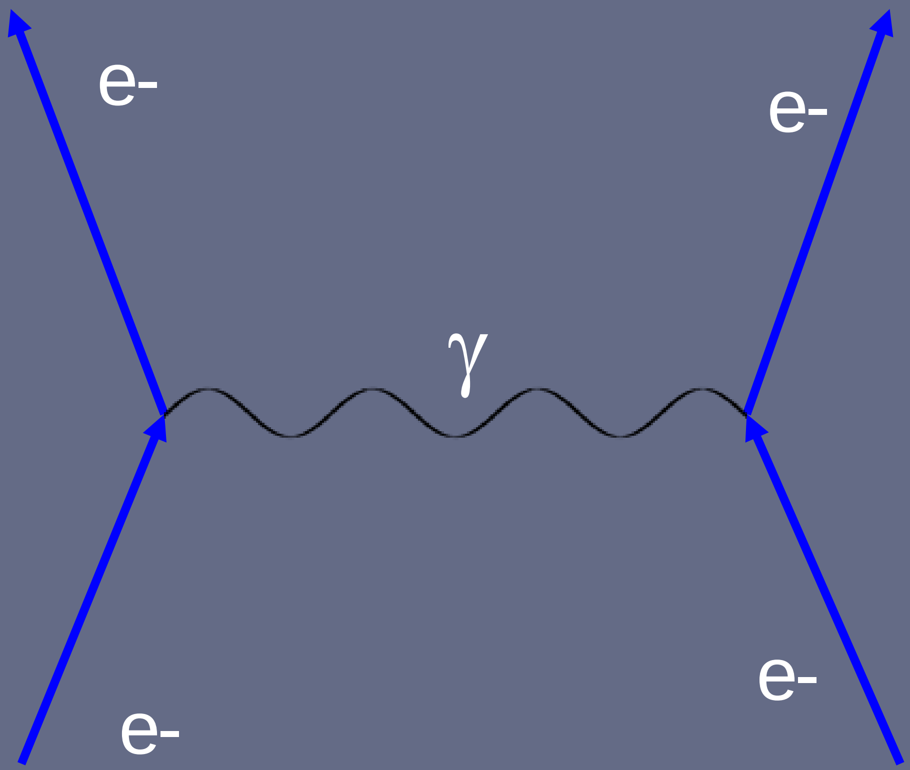
$$-ie(W_\mu^+ A_\nu - W_\nu^+ A_\mu) + ig' c_w (W_\mu^+ Z_\nu - W_\nu^+ Z_\mu|^2 +$$

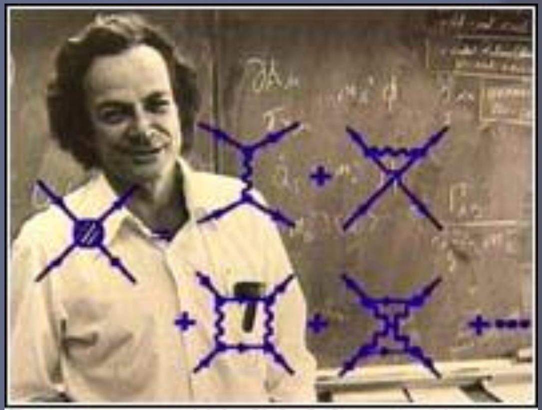
$$-\frac{1}{4} |\partial_\mu Z_\nu - \partial_\nu Z_\mu + ig' c_w (W_\mu^- W_\nu^+ - W_\mu^+ W_\nu^-)|^2 +$$

$$-\frac{1}{2} M_\eta^2 \eta^2 - \frac{g M_\eta^2}{8 M_W} \eta^3 - \frac{g'^2 M_\eta^2}{32 M_W} \eta^4 + |M_W W_\mu^+ + \frac{g}{2} \eta W_\mu^+|^2 +$$

$$+\frac{1}{2} |\partial_\mu \eta + i M_Z Z_\mu + \frac{ig}{2c_w} \eta Z_\mu|^2 - \sum_f \frac{g}{2} \frac{m_f}{M_W} \bar{\Psi}_f \Psi_f \eta$$

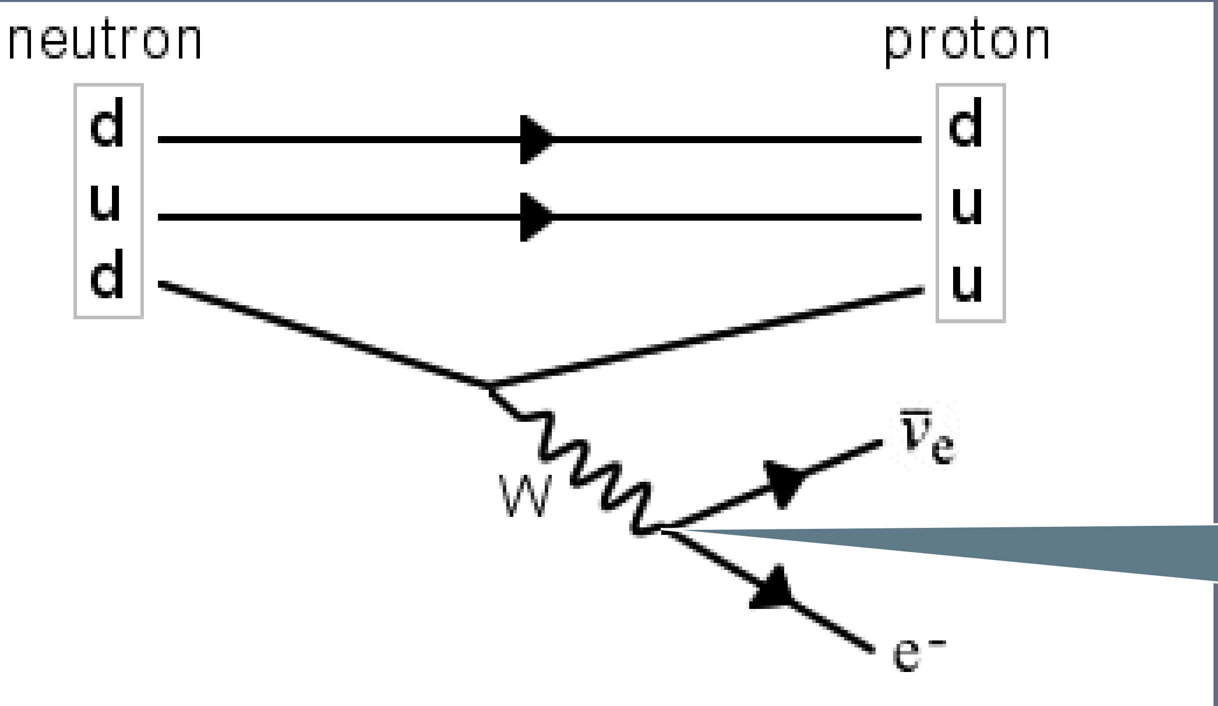
Feynman – diagrame și reguli





Feynman earned his Nobel for creating these diagrams
(Courtesy Auckland University)

Feynman – diagrame și reguli



Beta radioactive decay via the weak interaction

Sarcina electrică se conservă la fiecare vertex

În viziunea unui artist:
Cuarc bottom interacționând cu Higgs

Sursa: Sized Matter-Perception of the Extreme Unseen, Jan-Henrik Andersen

4. $$\boxed{F_x[\phi] = \phi(x)}$$

$$\frac{\delta F_x[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\phi(x) + \epsilon\delta(x-y) - \phi(x)\Big) = \delta(x-y)$$

5. $$F_x[\phi] = \nabla_x\phi(x)$$

$$\frac{\delta F_x[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\nabla_x\big(\phi(x) + \epsilon\delta(x-y)\big) - \nabla_x\phi(x)\Big) = \nabla_x\delta(x-y)$$

Relativitatea specială, principii, particule, cîmpuri

① t' Hooft : Relativitatea specială este teoria ce afirmă că spațiul și timpul manifestă „a particular symmetry pattern"

(i) There is a transformation law, and these transformations form a group

(ii) Consider a system in which a set of physical variables is described as being a correct solution to the laws of physics. Then if all these physical variables are transformed appropriately according to the given transformation law, one obtain a new solution to the laws of physics.

② Landau : ⓐ principiul relativității : „legile fizicii sînt identice în toate sistemele de referință inerțiale" sau „ecuațiile ce descriu legile naturii sînt invariante în raport transformările de coordonate și timp la trecerea de la un sistem de referință la altul"

ⓑ viteza de propagare a unui semnal ~~luminos~~ este aceeași în toate sistemele de referință inerțiale - viteza luminii

3. Eveniment, interval spaţio-temporal

Eveniment un punct în spaţiu-timp

„punct de univers"

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

- intervalul între două evenimente

$$s_{12} = \left[c^2(t_2-t_1)^2 - (x_2-x_1)^2 - (y_2-y_1)^2 - (z_2-z_1)^2 \right]^{1/2}$$

or

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

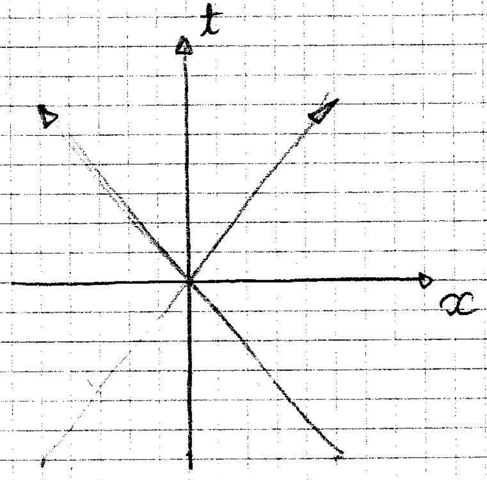
* Intervalul între două evenimente este acelaşi în toate sistemele de referinţă inerţiale, adică un invariant la transformarea de la un sistem de referinţă la altul

AT

CONSECINTA A CONSTANTEI VITEZEI LUMINII

- interval temporal $s_{12}^2 > 0$
- interval spatial $s_{12}^2 < 0$

Obs: natura intervalului nu depinde de sistemul de referinţă inerţial ales



(4) Timpul propriu

* Timpul indicat de un ceasornic legat rigid de corpul studiat !

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2 = c^2\,dt'^2 \Rightarrow dt' = \frac{ds}{c} = \frac{1}{c}\sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}$$

sau

$$dt' = dt\sqrt{1 - \frac{v^2}{c^2}}$$

$$\boxed{dt' < dt}$$

Timpul propriu ~~indicat~~ al unui corp în mișcare este totdeauna mai mic decât timpul corespunzător în referențialul fix.

$t \rightarrow$ timpul în SRI fix

$t' \rightarrow$ timpul în SRI legat de corp

de

lerut

Identify what factorize $\omega_{\mu\nu}$ and ϵ_μ on both sides:

$$\frac{i}{2}\omega_{\mu\nu} D(\Lambda,a) J^{\mu\nu} D^{-1}(\Lambda,a) = \frac{i}{2}\omega_{\rho\gamma} \Lambda_\alpha{}^\rho \Lambda_\beta{}^\sigma J^{\alpha\beta} + i\,\omega_{\sigma\theta} \Lambda_\alpha{}^\sigma \Lambda_\xi{}^\theta a^\xi P^\alpha$$

$$\omega_{\mu\nu} D(\Lambda,a) J^{\mu\nu} D^{-1}(\Lambda,a) = \omega_{\mu\nu} \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu J^{\alpha\beta} + (\omega_{\sigma\theta} - \omega_{\theta\sigma}) \Lambda_\alpha{}^\sigma \Lambda_\xi{}^\theta a^\xi P^\alpha$$

$$\omega_{\mu\nu} \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu a^\beta P^\alpha$$

$$- \omega_{\mu\nu} \Lambda_\alpha{}^\nu \Lambda_\beta{}^\mu a^\beta P^\alpha = -\omega_{\mu\nu} \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu a^\alpha P^\beta$$

$\Rightarrow$

$$\boxed{D(\Lambda,a) J^{\mu\nu} D^{-1}(\Lambda,a) = \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu \left(J^{\alpha\beta} + a^\beta P^\alpha - a^\alpha P^\beta \right)}$$

Look at ϵ_μ

$$-i\epsilon_\mu D(\Lambda,a) P^\mu D^{-1}(\Lambda,a) = -i \Lambda_\alpha{}^\sigma \epsilon_\sigma P^\alpha \quad \Rightarrow$$

$$\boxed{D(\Lambda,a) P^\mu D^{-1}(\Lambda,a) = \Lambda_\alpha{}^\mu P^\alpha}$$

<u>We can obtain now the commutation relations</u>

In the last two transformation ~~rul~~ laws of $J^{\mu\nu}$ and P^μ make

$$D(\Lambda,a) = D(1+\omega,\epsilon) \quad \Rightarrow \quad D^{-1}(\Lambda,a) = D(1-\omega, -(1-\omega)\epsilon) \simeq D(1-\omega,-\epsilon)$$

since $D^{-1}(\Lambda,a) = D(\Lambda^{-1}, -\Lambda^{-1}a)$

Therefore expanding

$$D(1+\omega,\epsilon) = I + \frac{1}{2}i\omega_{\mu\nu} J^{\mu\nu} - i\epsilon_\mu P^\mu$$

$$D(1-\omega,-\epsilon) = I - \frac{1}{2}i\omega_{\mu\nu} J^{\mu\nu} + i\epsilon_\mu P^\mu$$

$$\Lambda_\alpha{}^\mu = \delta^\mu_\alpha - \omega^\mu{}_\alpha \; ; \quad \Lambda_\beta{}^\nu = \delta^\nu_\beta - \omega^\nu{}_\beta$$

obtain

$$\left(1 + \frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu} - i\epsilon_\mu P^\mu\right) J^{\rho\sigma} \left(1 - \frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu} + i\epsilon_\mu P^\mu\right) = (\delta^\rho_\alpha - \omega^\rho{}_\alpha)(\delta^\sigma_\beta - \omega^\sigma{}_\beta)(J^{\alpha\beta} + \epsilon^\beta P^\alpha - \epsilon^\alpha P^\beta)$$

and begin the identification — see previous calculations (two pages before)

$$J^{\rho\sigma} + \frac{1}{2}i\omega_{\mu\nu}[J^{\mu\nu}, J^{\rho\sigma}] - i\epsilon_\mu [P^\mu, J^{\rho\sigma}] = J^{\rho\sigma} - \delta^\rho_\alpha \omega^\sigma{}_\beta J^{\alpha\beta} - \delta^\sigma_\beta \omega^\rho{}_\alpha J^{\alpha\beta} + \delta^\rho_\alpha \delta^\sigma_\beta (\epsilon^\beta P^\alpha - \epsilon^\alpha P^\beta)$$

so concluding we have

$$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho}J^{\mu\sigma} + \eta^{\nu\sigma}J^{\rho\mu} - \eta^{\rho\mu}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu}$$

$$-i\epsilon_\mu[P^\mu, J^{\rho\sigma}] = \epsilon^\sigma P^\rho - \epsilon^\rho P^\sigma \Rightarrow -i\epsilon_\mu[P^\mu, J^{\rho\sigma}] = \eta^{\sigma\mu}\epsilon_\mu P^\rho - \eta^{\rho\mu}\epsilon_\mu P^\sigma \text{ so}$$

$$\boxed{i[P^\mu, J^{\rho\sigma}] = \eta^{\rho\mu}P^\sigma - \eta^{\sigma\mu}P^\rho}$$

Analogously for $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$ $\quad a = \epsilon^\mu$

$$D(\Lambda,a)\, P^\mu\, D^{-1}(\Lambda,a) \Rightarrow$$

$$(1 - i\epsilon_\alpha P^\alpha)\, P^\mu\, (1 + i\epsilon_\alpha P^\alpha) = \delta^\mu{}_\gamma P^\gamma$$

$$\cancel{P^\mu} + i\epsilon_\alpha[P^\mu, P^\alpha] = \cancel{P^\mu} \Rightarrow \boxed{[P^\mu, P^\nu] = 0}$$

Observation: from $g_{\alpha\beta} \dfrac{\partial x'^{\alpha}}{\partial x^{\gamma}} \dfrac{\partial x'^{\beta}}{\partial x^{\delta}} = g_{\gamma\delta}$ or $g_{\alpha\beta} \Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} = g_{\gamma\delta}$ introduce the matrices

g such that $(g)_{\mu\nu} = g_{\mu\nu}$

Λ $\quad (\Lambda)_{\alpha\gamma} = \Lambda^{\alpha}{}_{\gamma} \Rightarrow (\Lambda^T)_{\alpha\gamma} = (\Lambda)_{\gamma\alpha} = \Lambda^{\gamma}{}_{\alpha}$

also $\Lambda^{\alpha}{}_{\gamma} = (\Lambda)_{\alpha\gamma} = (\Lambda^T)_{\gamma\alpha}$

and therefore the basic property can be written as a matrix equation $\Lambda^T g \Lambda = g$

I

$$\Lambda^T g \Lambda = g \Rightarrow \det g \cdot (\det \Lambda)^2 = \det g \Rightarrow (\det\Lambda)^2 = 1 \Rightarrow \boxed{\det \Lambda = \pm 1}$$

II

$$g_{\alpha\beta} \Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} = g_{\gamma\delta} \xRightarrow{\gamma=\delta=0} (-1)(\Lambda^0{}_0)^2 + \sum_{i=1}^{3} (\Lambda^i{}_0)^2 = -1 \Rightarrow (\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^{3} (\Lambda^i{}_0)^2 \geq 1$$

(proper Lorentz transformations) $\quad \Lambda^0{}_0 \geq 1$, $\det\Lambda = 1$

$\Lambda^0{}_0 \geq 1$, $\det \Lambda = -1$ (space inversion)

$\Lambda^0{}_0 \leq -1$, $\det \Lambda = 1$

$\Lambda^0{}_0 \leq -1$, $\det \Lambda = -1$ (time reversal)

The subgroup of the rotations

$\Lambda^i{}_j = R_{ij}$; $\Lambda^0{}_0 = 1$; $\Lambda^i{}_0 = \Lambda^0{}_i = 0$; Here R_{ij} is a unimodular orthogonal matrix $\boxed{\begin{cases} \det R = 1 \\ R^T R = 1 \end{cases}}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}$$

The boosts transformations : change the velocity of coordinate frame

one observer O → sees a particle at rest

a second observer O' → sees it moving with $\vec{v}$

QUESTION : Find the corresponding Lorentz transformation between the two observers

$$\left.\begin{array}{l} dx'^\alpha = \Lambda^\alpha{}_\beta \, dx^\beta \\ d\vec{x} = 0 \, (dx^1 = dx^2 = dx^3 = 0) \end{array}\right\} \Rightarrow \begin{cases} dx'^i = \Lambda^i{}_0 \, dt & i = 1,2,3 \\ dt' = \Lambda^0{}_0 \, dt \end{cases}$$

But $\boxed{\dfrac{dx'^i}{dt'} = v_i}$ (the i component of velocity) $\Rightarrow$ $\Lambda^i{}_0 = v_i \Lambda^0{}_0$ for $i = 1,2,3$

Also from the fundamental property of Lorentz transf $g_{\alpha\beta} \Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta = g_{\gamma\delta}$ for $\gamma = \delta = 0$ we obtain

$$-1 = \Lambda^\alpha{}_0 \Lambda^\beta{}_0 \, g_{\alpha\beta} = \sum_{i=1,2,3} (\Lambda^i{}_0)^2 - (\Lambda^0{}_0)^2$$

So we have

$$\begin{cases} \Lambda^i{}_0 = v_i \Lambda^0{}_0 \\ \sum_{i=1,2,3} (\Lambda^i{}_0)^2 - (\Lambda^0{}_0)^2 = -1 \end{cases} \Rightarrow (\Lambda^0{}_0)^2 (v^2 - 1) = -1 \Rightarrow \begin{cases} \Lambda^0{}_0 = \dfrac{1}{\sqrt{1 - v^2}} \equiv \gamma \\ \Lambda^i{}_0 = v_i \gamma \end{cases}$$

Obs: The other $\Lambda^\alpha{}_\beta$ i.e. $\Lambda^i{}_j$ are not uniquely determined, because if $\Lambda^\alpha{}_\beta$ carries a particle from rest to velocity $\vec{v}$, then so does $\Lambda^\alpha{}_\gamma R^\gamma{}_\beta$ where R is an arbitrary rotation. One choice, satisfying the fundamental relations is:

$$\Lambda^i{}_j = \delta_{ij} + v_i v_j \frac{\gamma - 1}{v^2} \; ; \; \Lambda^0{}_j = \gamma v_j$$

$$\begin{pmatrix} \gamma & v_1 \gamma & v_2 \gamma & v_3 \gamma \\ \gamma v_1 & \frac{v_1^2 (\gamma - 1)}{v^2} + 1 & \frac{v_1 v_2 (\gamma - 1)}{v^2} & \frac{v_1 v_3 (\gamma - 1)}{v^2} \\ \gamma v_2 & & \frac{v_2^2 (\gamma - 1)}{v^2} + 1 & \frac{v_2 v_3 (\gamma - 1)}{v^2} \\ \gamma v_3 & & & \frac{v_3^2 (\gamma - 1)}{v^2} + 1 \end{pmatrix}$$

ABOUT VECTORS AND TENSORS

Let us consider the Lorentz transformation $x'^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta}$

The inverse:

From $\Lambda^{T} g \Lambda = g \Rightarrow \Lambda^{-1} = g^{-1} \Lambda^{T} g = g \Lambda^{T} g$ so in matrix form

$$(\Lambda^{-1})_{\mu\nu} = (g^{-1})_{\mu\sigma} (\Lambda^{T})_{\sigma\rho} (g)_{\rho\nu} \quad \text{or}$$

$$\Lambda_{\nu}{}^{\mu} \equiv (\Lambda^{-1})^{\mu}{}_{\nu} = g_{\nu\rho} g^{\mu\sigma} \Lambda^{\rho}{}_{\sigma}$$

$$\boxed{\Lambda_{\nu}{}^{\mu} \equiv g_{\nu\rho} g^{\mu\sigma} \Lambda^{\rho}{}_{\sigma}}$$

Let us observe that if $\boxed{x'^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta}} \longrightarrow$

$$\Lambda_{\alpha}{}^{\gamma} x'^{\alpha} = \Lambda_{\alpha}{}^{\gamma} \Lambda^{\alpha}{}_{\beta} x^{\beta}$$

and since $\Lambda_{\alpha}{}^{\gamma} \Lambda^{\alpha}{}_{\beta} = g_{\alpha\rho} g^{\gamma\sigma} \Lambda^{\rho}{}_{\sigma} \Lambda^{\alpha}{}_{\beta} =$

$$= g_{\sigma\beta} g^{\gamma\sigma} = \delta^{\gamma}{}_{\beta}$$

we have

$$\Lambda_{\alpha}{}^{\gamma} x'^{\alpha} = \delta^{\gamma}{}_{\beta} x^{\beta} \quad \text{or}$$

THEN $\longrightarrow$ $\boxed{x^{\gamma} = \Lambda_{\alpha}{}^{\gamma} x'^{\alpha}}$

Contravariant four-vector:

~~Orice cov~~ Is a quantity which at a Lorentz transformation $x'^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta}$ transforms as dx^{α}, i.e. undergoes the transformation:

$$\boxed{V^{\alpha\prime} = \Lambda^{\alpha}{}_{\beta} V^{\beta}}$$

Covariant four-vector

Is a quantity U_{α} whose transformation rule is

$$\boxed{U'_{\alpha} = \Lambda_{\alpha}{}^{\beta} U_{\beta}}$$

when the coord syst. transform as $x'^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta}$ i.e. it transforms with the inverse matrix

Observations

1) To every contravariant 4-vector, V^α there corresponds a covariant four-vector

$$V_\alpha \equiv g_{\alpha\beta} V^\beta$$

Indeed is to be observed that:

$$V'_\alpha = g_{\alpha\beta} V'^\beta = g_{\alpha\beta} \Lambda^\beta{}_\gamma V^\gamma = g_{\alpha\beta} g^{\gamma\delta} \Lambda^\beta{}_\gamma V_\delta = \Lambda_\alpha{}^\delta V_\delta$$

i.e. V_α transforms as a covariant vector

2) To every covariant 4-vector, U_α, there corresponds a contravariant vector

$$U^\alpha = g^{\alpha\beta} U_\beta$$

3) $$\frac{\partial}{\partial x'^\alpha} = \Lambda_\alpha{}^\beta \frac{\partial}{\partial x^\beta}$$ i.e. the gradient (the derivative in respect to contravariant vector x^β) is a covariant vector.

Proof $$\frac{\partial}{\partial x'^\alpha} = \frac{\partial}{\partial x^\beta} \frac{\partial x^\beta}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial}{\partial x^\beta}$$

Now, since $$x^\beta = \Lambda_\alpha{}^\beta x'^\alpha \Rightarrow \frac{\partial x^\beta}{\partial x'^\alpha} = \Lambda_\alpha{}^\beta$$

so $$\frac{\partial}{\partial x'^\alpha} = \Lambda_\alpha{}^\beta \frac{\partial}{\partial x^\beta}$$

q.e.d.

IULIE 2005

LORENTZ GROUP ALGEBRA

diag$(-1,1,1,1)$

- Consider first the transformation of a contravariant vector x^μ

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$ Here $\Lambda^\mu{}_\nu$ stands for $(\Lambda)_{\mu\nu}$ in matricial notation

The Lorentz transformations leaves invariant the four-interval $x^\mu x_\mu$. So

$$x'^\mu x'_\mu = x^\nu x_\nu$$

But $x'^\mu x'_\mu = \Lambda^\mu{}_\nu x^\nu g_{\mu\sigma} x'^\sigma = \Lambda^\mu{}_\nu x^\nu g_{\mu\sigma} \Lambda^\sigma{}_\rho x^\rho = \Lambda^\mu{}_\nu \Lambda^\sigma{}_\rho g_{\mu\sigma} x^\nu x^\rho$

So

$$g_{\mu\sigma} \Lambda^\mu{}_\nu \Lambda^\sigma{}_\rho \underline{\underline{x^\nu x^\rho}} = \underline{x}^\nu g_{\nu\rho} \underline{x}^\rho$$ so relabelling for some indexes

initially $(g_{\mu\sigma} \Lambda^\mu{}_\nu \Lambda^\sigma{}_\rho = g_{\nu\rho})$

The basic property of Lorentz transformation

$$\boxed{g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}}$$

Obs: Poate fi văzută și ca invarianța tensorului metric la transf. Lorentz?

(encript the fundamental principles of special relativity: relativity principle and constancy of light velocity)

Tensor metric: observations $(g)_{\mu\nu} = g_{\mu\nu}$

$$(g^{-1})_{\mu\nu} = g^{\mu\nu}$$

$$g_{\mu\nu} = g_{\nu\mu}$$

In matrix notation

$$\boxed{\Lambda^T g \Lambda = g}$$

$$(\Lambda^T)_{\mu\rho} (g)_{\rho\beta} (\Lambda)_{\beta\gamma} = (g)_{\mu\gamma}$$

$$\parallel$$

$$(\Lambda)_{\rho\mu}$$

$$\parallel$$

$$\Lambda^\rho{}_\mu g_{\rho\beta} \Lambda^\beta{}_\gamma = g_{\mu\gamma}$$

- The inverse transformation Λ^{-1} can be expressed using the relation $\Lambda^T g \Lambda = g$

$$\Rightarrow \Lambda^{-1} = g^{-1} \Lambda^T g \; (= g \Lambda^T g)$$

Obs Weinberg is using $(\Lambda^{-1})_{\mu\nu} = \Lambda_\nu{}^\mu$

So if $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ define $x^\mu = x'^\nu \Lambda_\nu{}^\mu = \Lambda_\nu{}^\mu x'^\nu = ((\Lambda^{-1})_{\mu\nu} (x$

Obviously $x^\mu = \Lambda_\nu{}^\mu x'^\nu = \Lambda_\nu{}^\mu x^\beta \Lambda^\nu{}_\beta = \Lambda_\nu{}^\mu \Lambda^\nu{}_\beta x^\beta$

$$\Rightarrow \delta^\mu{}_\beta x^\beta = \Lambda_\nu{}^\mu \Lambda^\nu{}_\beta x^\beta \Rightarrow \boxed{\Lambda_\nu{}^\mu \Lambda^\nu{}_\beta = \delta^\mu{}_\beta}$$

$|||$ in matrix notation

$$\Lambda^{-1}\Lambda = I \Leftarrow (\Lambda^{-1})_{\mu\nu} (\Lambda)_{\nu\beta} = (\delta)_{\mu\beta}$$

Now $\Lambda^{-1} = g^{-1} \Lambda^T g \Rightarrow$

$$(\Lambda^{-1})_{\mu\nu} = (g^{-1})_{\mu\sigma} (\Lambda^T)_{\sigma\rho} (g)_{\rho\nu} \Rightarrow$$

$$\Rightarrow \boxed{\Lambda_\nu{}^\mu = g_{\nu\rho} g^{\mu\sigma} \Lambda^\rho{}_\sigma}$$

Then verify that

$$\Lambda_\nu{}^\mu \Lambda^\nu{}_\rho = \underbrace{\eta_{\beta\nu}\, \eta^{\mu\sigma}\, \Lambda^\beta{}_\sigma \Lambda^\nu{}_\rho}_{\text{Fundam. Property of Lorentz transf}} = \eta^{\mu\sigma}\eta_{\sigma\rho} = \delta^\mu{}_\rho \quad (\text{OK } q.e.d.)$$

- The generators of Lorentz group

Consider a L. transformation differing infinitesimally from identity

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

Then a property of $\omega^\mu{}_\nu$ obtained from the fundamental property

$$\eta_{\alpha\beta}\Lambda^\alpha{}_\gamma\Lambda^\beta{}_\rho = \eta_{\gamma\rho} \Rightarrow \eta_{\alpha\beta}\left(\delta^\alpha{}_\gamma + \omega^\alpha{}_\gamma\right)\left(\delta^\beta{}_\rho + \omega^\beta{}_\rho\right) = \eta_{\gamma\rho}$$

to first order in ω

$$\underbrace{\eta_{\alpha\beta}\delta^\alpha{}_\gamma\delta^\beta{}_\rho}_{} + \eta_{\alpha\beta}\omega^\alpha{}_\gamma\delta^\beta{}_\rho + \eta_{\alpha\beta}\delta^\alpha{}_\gamma\omega^\beta{}_\rho = \eta_{\gamma\rho}$$

$$\eta_{\gamma\rho} + \omega_{\rho\gamma} + \omega_{\gamma\rho} = \eta_{\gamma\rho} \Rightarrow \boxed{\omega_{\rho\gamma} = -\omega_{\gamma\rho}}$$

ω is an antisymmetric matrix $\quad 16 - 4 - \frac{12}{2} = 6$ independent parameters

If $D(\Lambda)$ is a Lorentz group representation it will depend on six parameters. Since a group representation generator is defined as $M = \left.\frac{\partial D(x)}{\partial x}\right|_{x=0}$

In cazul nostru putem scrie

$$D(1+\omega) = I + \frac{1}{2} i\, \omega_{\mu\nu} J^{\mu\nu} + \dots$$

(deci derivata matricii reprezentarii in raport cu parametrul in consideratie, x, in $x=0$ consider că este reprezentată identitatea $\underbrace{D(0,0,\dots)}_{n \text{ parametri}} = I$)

Evident, din $\omega_{\mu\nu} = -\omega_{\nu\mu} \Rightarrow J^{\mu\nu} = -J^{\nu\mu}$

Using group property of Lorentz transformations and of corresponding representations we can obtain transformation behaviour of the generators $J^{\mu\nu}$ and their commutation relations

Let consider $D(\Lambda)$, $D(1+\omega)$ and $D(\Lambda^{-1})$. Then

$$D(\Lambda)D(1+\omega)D(\Lambda^{-1}) = D(\Lambda(1+\omega)\Lambda^{-1})$$ (property of group representations)

Now use the previous expansion and obtain

$$D(\Lambda)\left(I+\frac{1}{2}i\omega_{\rho\sigma}J^{\rho\sigma}\right)D(\Lambda^{-1}) = D(1+\Lambda\omega\Lambda^{-1})$$

we also expand this expression

$$I+\frac{1}{2}i\,\omega_{\rho\sigma}D(\Lambda)J^{\rho\sigma}D(\Lambda^{-1}) = I+\frac{1}{2}i(\Lambda\omega\Lambda^{-1})_{\mu\nu}J^{\mu\nu}$$

Obtain the expression $(\Lambda\omega\Lambda^{-1})_{\mu\nu}$

$$(\Lambda\omega\Lambda^{-1})_{\mu\nu}J^{\mu\nu} = \eta_{\mu\gamma}\Lambda^{\gamma}{}_{\alpha}\omega^{\alpha}{}_{\beta}\Lambda_{\nu}{}^{\beta}J^{\mu\nu} = \eta_{\mu\gamma}\Lambda^{\gamma}{}_{\alpha}\eta^{\alpha\rho}\omega_{\rho\sigma}\Lambda_{\nu}{}^{\sigma}J^{\mu\nu}$$

$$= \omega_{\rho\sigma}\underbrace{\eta_{\mu\gamma}\eta^{\alpha\rho}\Lambda^{\gamma}{}_{\alpha}}\Lambda_{\nu}{}^{\sigma}J^{\mu\nu}$$

$$\Lambda_{\mu}{}^{\rho}\Lambda_{\nu}{}^{\sigma}J^{\mu\nu}$$

so

$$\boxed{D(\Lambda)J^{\rho\sigma}D(\Lambda^{-1}) = \Lambda_{\mu}{}^{\rho}\Lambda_{\nu}{}^{\sigma}J^{\mu\nu}}$$

If consider starting from this relation also $D(\Lambda)$ infinitesimal we can arrive at comutation relations between generators

Obs if $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}+\omega^{\mu}{}_{\nu}$ then $\Lambda_{\nu}{}^{\mu} = \delta^{\mu}{}_{\nu}-\omega^{\mu}{}_{\nu}$!

Proof $\Lambda_{\nu}{}^{\mu}\Lambda^{\nu}{}_{\beta} = (\delta^{\mu}{}_{\nu}-\omega^{\mu}{}_{\nu})(\delta^{\nu}{}_{\beta}+\omega^{\nu}{}_{\beta}) \simeq \delta^{\mu}{}_{\beta}-\omega^{\mu}{}_{\beta}+\omega^{\mu}{}_{\beta} = \delta^{\mu}{}_{\beta}$

or $\Lambda_{\nu}{}^{\mu} = \eta_{\nu\alpha}\eta^{\mu\beta}\Lambda^{\alpha}{}_{\beta} = \eta_{\nu\alpha}\eta^{\mu\beta}(\delta^{\alpha}{}_{\beta}+\omega^{\alpha}{}_{\beta}) =$

$= \eta_{\nu\beta}\eta^{\mu\beta}+\eta_{\nu\alpha}\eta^{\mu\beta}\omega^{\alpha}{}_{\beta} = \delta^{\mu}{}_{\nu}+\eta^{\mu\beta}\omega_{\nu\beta}$

$= \delta^{\mu}{}_{\nu}-\eta^{\mu\beta}\omega_{\beta\nu} = \delta^{\mu}{}_{\nu}-\omega^{\mu}{}_{\nu}$

Now

$$\left(I+\frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu}\right)J^{\rho\sigma}\left(1-\frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu}\right) = (\delta^{\rho}{}_{\mu}-\omega^{\rho}{}_{\mu})(\delta^{\sigma}{}_{\nu}-\omega^{\sigma}{}_{\nu})J^{\mu\nu} \Rightarrow$$

$$\Rightarrow \frac{1}{2}i\omega_{\mu\nu}[J^{\mu\nu},J^{\rho\sigma}] = -\delta^{\rho}_{\mu}\omega^{\sigma}{}_{\nu}J^{\mu\nu}-\delta^{\sigma}_{\nu}\omega^{\rho}{}_{\mu}J^{\mu\nu}$$ or

$$\frac{1}{2}i\omega_{\mu\nu}[J^{\mu\nu},J^{\rho\sigma}] = -\omega^{\sigma}{}_{\nu}J^{\rho\nu}-\omega^{\rho}{}_{\mu}J^{\mu\sigma}$$

Try to figure out also on right side $\omega_{\dots}$. Use lower to metric and obtain

$$\frac{1}{2} i\omega_{\mu\nu}[J^{\mu\nu}, J^{\rho\sigma}] = -\eta^{\sigma\beta}\omega_{\beta\nu}J^{\rho\nu} - \eta^{\rho\alpha}\omega_{\alpha\mu}J^{\mu\sigma}$$

or

$$i\omega_{\mu\nu}[J^{\mu\nu}, J^{\rho\sigma}] = -\underbrace{2\omega_{\beta\nu}}_{\omega_{\beta\nu}-\omega_{\nu\beta}}\eta^{\sigma\beta}J^{\rho\nu} - \underbrace{2\omega_{\alpha\mu}}_{\omega_{\alpha\mu}-\omega_{\mu\alpha}}\eta^{\rho\alpha}J^{\mu\sigma}$$

so

$$i\omega_{\mu\nu}[J^{\mu\nu}, J^{\rho\sigma}] = \omega_{\underset{\mu}{\nu}\underset{\nu}{\beta}}\eta^{\sigma\beta}J^{\rho\nu} - \omega_{\underset{\mu}{\beta}\underset{\nu}{\nu}}\eta^{\sigma\beta}J^{\rho\nu} + \omega_{\underset{\mu}{\mu}\underset{\nu}{\alpha}}\eta^{\rho\alpha}J^{\mu\sigma} - \omega_{\underset{\mu}{\alpha}\underset{\nu}{\mu}}\eta^{\rho\alpha}J^{\mu\sigma}$$

so we obtain

$$i\omega_{\mu\nu}[J^{\mu\nu}, J^{\rho\sigma}] = \omega_{\mu\nu}\eta^{\sigma\nu}J^{\rho\mu} - \omega_{\mu\nu}\eta^{\sigma\mu}J^{\rho\nu} + \omega_{\mu\nu}\eta^{\rho\nu}J^{\mu\sigma} - \omega_{\mu\nu}\eta^{\rho\mu}J^{\nu\sigma}$$

or (as in Weinberg)

$$\boxed{i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\rho\nu}J^{\mu\sigma} - \eta^{\rho\mu}J^{\nu\sigma} + \eta^{\sigma\nu}J^{\rho\mu} - \eta^{\sigma\mu}J^{\rho\nu}}$$

Cum să o rețin formula cu semnele corespunzătoare?

$\mu\nu\ \rho\sigma$ Formăm perechi posibile

$\mu\rho$	(−)	I + III
$\mu\sigma$	(+)	I + IV
$\nu\rho$	(+)	II + III
$\nu\sigma$	(−)	II IV

POINCARÉ GROUP ALGEBRA

- or inhomogeneous Lorentz group: include translations along four possible directions

$$x'^\mu = x^\mu + a^\mu$$

↳ translation along μ direction with a^μ

An element of Poincaré group (Λ, a)

$$\boxed{x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu} \quad \rightarrow \text{Definition}$$

- Composition of two transformations

$$x''^\mu = \Lambda_2{}^\mu{}_\nu x'^\nu + a_2^\mu = \Lambda_2{}^\mu{}_\nu \left(\Lambda_1{}^\nu{}_\beta x^\beta + a_1^\nu\right) + a_2^\mu = ;$$

$$= \underbrace{\Lambda_2{}^\mu{}_\nu \Lambda_1{}^\nu{}_\beta}_{\Lambda = \Lambda_2\Lambda_1} x^\beta + \underbrace{\Lambda_2{}^\mu{}_\nu a_1^\nu + a_2^\mu}_{\Lambda_2 a_1 + a_2}$$

So

$$\boxed{(\Lambda_2, a_2)\circ(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)}$$

- The inverse of a Poincaré transformation

If we want that $(\Lambda_2, a_2)\circ(\Lambda_1, a_1) = (I, 0) \Rightarrow$

$$\Lambda_2 = \Lambda_1^{-1},$$

and so

$$a_2 = -\Lambda_1^{-1} a_1$$

$\Rightarrow$ $\boxed{(\Lambda^{-1}; -\Lambda^{-1}a) \text{ is the inverse of } (\Lambda, a)}$

Now the group depends on 10 parameters $(\omega_{01}, \omega_{02}, \omega_{03}, \omega_{12}, \omega_{13}, \omega_{23}, a_0, a_1, a_2, a_3)$ and the group is characterized by ten generators $J^{\mu\nu}$ and P^μ

We follow a general discussion now using the properties of group representations and deduce how P^μ transform as well as the corresponding commutation relations

Let us remark

$$D(\Lambda, 0) \xrightarrow[\text{is}]{\text{inverse}} D(-\Lambda, 0)$$

$$D(0, a) \xrightarrow[\text{is}]{\text{inverse}} D(0, -a)$$

and $D(\Lambda, a) \xrightarrow[\text{is}]{\text{inverse}} D(\Lambda^{-1}, -\Lambda^{-1}a)$

Again consider first ... $D(1+\omega, \epsilon)$ ...
and expansion

$$D(1+\omega, \epsilon) = I + \frac{1}{2} i\omega_{\mu\nu} J^{\mu\nu} - i\epsilon_\mu P^\mu$$

recall the property considered previously, that

$$D(\Lambda,a)D(1+\omega,\epsilon)D^{-1}(\Lambda,a) = D(\Lambda(1+\omega), \Lambda\epsilon+a)\,D(\Lambda^{-1}, -\Lambda^{-1}a) =$$

$$= D(\Lambda(1+\omega)\Lambda^{-1}, -\Lambda(1+\omega)\Lambda^{-1}a + \Lambda\epsilon + a)$$

$$= D(\Lambda(1+\omega)\Lambda^{-1}, \cancel{-a} - \Lambda\omega\Lambda^{-1}a + \Lambda\epsilon \cancel{+a})$$

or finally

$$D(\Lambda,a)D(1+\omega,\epsilon)D^{-1}(\Lambda,a) = D(\Lambda(1+\omega)\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)$$

or again

$$D(\Lambda,a)D(1+\omega,\epsilon)D^{-1}(\Lambda,a) = D(1 + \underbrace{\Lambda\omega\Lambda^{-1}}_{\text{infinitesimal quantity}}, \underbrace{\Lambda\epsilon - \Lambda\omega\Lambda^{-1}a}_{\text{infinitesimal quantity}})$$

and consider the expansion

$$D(\Lambda,a)\left(I + \frac{1}{2} i\omega_{\mu\nu} J^{\mu\nu} - i\epsilon_\mu P^\mu\right)D^{-1}(\Lambda,a) = I + \frac{i}{2}(\Lambda\omega\Lambda^{-1})_{\mu\nu} J^{\mu\nu} - i(\Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)_\mu P^\mu$$

or

$$I + \frac{i}{2}\omega_{\mu\nu} D(\Lambda,a) J^{\mu\nu} D^{-1}(\Lambda,a) - i\epsilon_\mu D(\Lambda,a) P^\mu D^{-1}(\Lambda,a) =$$

$$= I + \frac{i}{2}(\Lambda\omega\Lambda^{-1})_{\mu\nu} J^{\mu\nu} - i(\Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)_\mu P^\mu$$

Some algebra:

$$(\Lambda\omega\Lambda^{-1})_{\alpha\beta} = \Lambda_\alpha{}^\rho \Lambda_\beta{}^\gamma \omega_{\rho\gamma}$$

$$(\Lambda\epsilon)_\alpha = \eta_{\alpha\rho}(\Lambda\epsilon)^\rho = \eta_{\alpha\rho}\Lambda^\rho{}_\theta \epsilon^\theta = \eta_{\alpha\rho}\eta^{\theta\sigma}\Lambda^\rho{}_\theta \epsilon_\sigma = \Lambda_\alpha{}^\sigma \epsilon_\sigma$$

$$(\Lambda\omega\Lambda^{-1}a)_\alpha = \eta_{\alpha\rho}(\Lambda\omega\Lambda^{-1}a)^\rho = \eta_{\alpha\rho}\Lambda^\rho{}_\gamma \omega^\gamma{}_\theta \Lambda_\xi{}^\theta a^\xi = \eta_{\alpha\rho}\eta^{\gamma\sigma}\Lambda^\rho{}_\gamma \omega_{\sigma\theta}\Lambda_\xi{}^\theta a^\xi$$

$$= \Lambda_\alpha{}^\sigma \Lambda_\xi{}^\theta \omega_{\sigma\theta} a^\xi$$

Consider the transformation of coordinates

$$x \longrightarrow x'$$

nonsingular:

$x'(x)$ and $x(x')$ are well-behaved differentiable functions so that the matrix $\dfrac{\partial x'^\alpha}{\partial x^\beta}$ has a well defined inverse $\dfrac{\partial x^\beta}{\partial x'^\alpha}$

THEOREM: The Lorentz transformations are the only nonsingular coordinate transformations $x \to x'$ that leave invariant the proper time $d\tau$

$$d\tau^2 = dt^2 - d\vec{x}^2 = -g_{\mu\nu}\, dx^\mu dx^\nu$$

Proof:

$$\left.\begin{aligned} d\tau'^2 &= -g_{\alpha\beta}\, dx'^\alpha dx'^\beta = -g_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} dx^\gamma dx^\delta \\ d\tau^2 &= -g_{\gamma\delta}\, dx^\gamma dx^\delta \end{aligned}\right\} \xRightarrow{\text{invariance of } d\tau} \boxed{g_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} = g_{\gamma\delta}}$$

If we differentiate with respect to $x^\rho \Rightarrow$

$$0 = g_{\alpha\beta} \frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\rho} \frac{\partial x'^\beta}{\partial x^\delta} + \cancel{g_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial^2 x'^\beta}{\partial x^\delta \partial x^\rho}}$$

If instead of γ we have ρ and instead of ρ we put γ obtain

$$0 = g_{\alpha\beta} \frac{\partial^2 x'^\alpha}{\partial x^\rho \partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} + g_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\rho} \frac{\partial^2 x'^\beta}{\partial x^\delta \partial x^\gamma}$$

Analogously if instead of δ we have ρ and instead of ρ we put δ we obtain

$$0 = \underbrace{g_{\alpha\beta} \frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\delta} \frac{\partial x'^\beta}{\partial x^\rho}}_{} + \cancel{g_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial^2 x'^\beta}{\partial x^\rho \partial x^\delta}} \quad \Big| \cdot (-1)$$

$$\| $$

$$g_{\beta\alpha} \frac{\partial^2 x'^\beta}{\partial x^\gamma \partial x^\delta} \frac{\partial x'^\alpha}{\partial x^\rho} = g_{\alpha\beta} \frac{\partial^2 x'^\beta}{\partial x^\gamma \partial x^\delta} \frac{\partial x'^\alpha}{\partial x^\rho}$$

Summing up the last three equalities from the cancelations of various term results:

$$2 g_{\alpha\beta} \frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\rho} \frac{\partial x'^\beta}{\partial x^\delta} = 0$$

For a given γ (fixed) this equation can be written as

$$[X^{(\gamma)}]_{\rho\alpha} \cdot [g]_{\alpha\beta} \cdot [M]_{\beta\delta} = 0 = [R]_{\rho\delta} \quad (\forall)\ \rho, \delta = \overline{0,3}$$

where the Lagrangian derivative of $\mathcal{L}$ with respect to ϕ^α (or Eulerian associated with the variation with respect to ϕ^α) is

4N

$$\mathcal{E}^\alpha = \frac{\partial \mathcal{L}}{\partial \phi^\alpha} - \mathcal{D}_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)} \right)$$

In conclusion the total variation of the action is:

$$\delta S \equiv S' - S = \int_{R'} d^4x' \mathcal{L}'(x', \phi'^\alpha, \partial_\mu \phi'^\alpha) - \int_R \mathcal{L}(x, \phi^\alpha, \partial_\mu \phi^\alpha) d^4x = \int_{R'} d^4x' \mathcal{L}(x', \phi'^\alpha, \partial_\mu \phi'^\alpha) - \int_R d^4x \mathcal{L}(x, \phi, \partial_\mu \phi^\alpha)$$

$$= \int_R d^4x \left\{ \mathcal{D}_\mu \left(\mathcal{L} \delta x^\mu + \pi^\mu_\alpha \delta\phi^\alpha \right) + \mathcal{E}^\alpha \delta \phi^\alpha \right\}$$

with $\pi^\mu_\alpha = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)}$

Field equations (Euler-Lagrange equations for fields)

The equations of motion for fields can be derived from Hamilton principle. According to the assumptions involved in this principle, we restrict ourselves to special types of variations in which

(a) the region of integration is unchanged

(b) the variations of the state functions vanish identically over the boundary region R

Obs: For this special type of variation

$$\delta S = \int_R d^4x \, \mathcal{E}^\alpha \delta' \phi^\alpha$$

Hamilton principle requires that the expression δS must vanish identically for every choice of the region of integration and for every choice of the variational functions $\delta' \phi^\alpha$ (subject only to above restrictions).

It follows from this assumption that for each α

$$\mathcal{E}^\alpha = 0 \quad \text{or}$$

$$\frac{\partial \mathcal{L}}{\partial \phi^\alpha} - \mathcal{D}_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)} \right) = 0$$

i.e. the fields obeying the Euler-Lagrange equations make the action an extremum

Symmetry transformations

5N

A group of infinitesimal transformations depending on p parameters ϵ^k

$$\delta x^\mu = \Gamma^\mu_k(x)\,\epsilon^k \qquad \Gamma^\mu_k \to \text{group generators}$$

$$\delta' \phi^\alpha = G^\alpha_k(x,\phi)\,\epsilon^k \qquad G^\alpha_k \to \text{representations of these generators acting on fields } \phi^\alpha$$

which leave the action S invariant are called symmetry transformations

We distinguish between

INTERNAL SYMMETRIES : x^μ are unchanged, i.e. $\delta x^\mu = 0$
G^α_k are independent of x i.e. $G^\alpha_k = G^\alpha_k(\phi)$

SPACE-TIME SYMMETRIES : if the change of field variables is induced by a coordinate transformation, i.e. $\delta x^\mu \neq 0$

Let us observe that the variation of the action can be expressed in terms of total variations of the fields since $\delta\phi^\alpha = \delta'\phi^\alpha - \partial_\nu \phi^\alpha \delta x^\nu$

$$\Delta S = \int_R d^4x \left[\mathcal{D}_\mu \left(\mathcal{L}\,\delta x^\mu + \pi^\mu_\alpha \delta\phi^\alpha \right) + \mathcal{E}^\alpha \delta\phi^\alpha \right] = \int_R d^4x\, \mathcal{D}_\mu \left(\mathcal{L}\,\delta x^\mu + \pi^\mu_\alpha \delta\phi^\alpha \right) + \int_R d^4x\, \mathcal{E}^\alpha \delta\phi^\alpha =$$

$$= \int_{\Sigma_2} d\sigma_\mu \left(\mathcal{L}\,\delta x^\mu + \pi^\mu_\alpha \delta\phi^\alpha \right) - \int_{\Sigma_1} d\sigma_\mu \left(\mathcal{L}\,\delta x^\mu + \pi^\mu_\alpha \delta\phi^\alpha \right) + \int_R d^4x\, \mathcal{E}^\alpha \delta\phi^\alpha =$$

$$= \int_{\Sigma_2} d\sigma_\mu \left(\mathcal{L}\,\delta^\mu_\nu \delta x^\nu - \pi^\mu_\alpha \partial_\nu \phi^\alpha \delta x^\nu + \pi^\mu_\alpha \delta'\phi^\alpha \right) - \int_{\Sigma_1} d\sigma_\mu \left(\mathcal{L}\,\delta^\mu_\nu \delta x^\nu - \pi^\mu_\alpha \partial_\nu \phi^\alpha \delta x^\nu + \pi^\mu_\alpha \delta'\phi^\alpha \right) + \int_R d^4x\, \mathcal{E}^\alpha \delta\phi^\alpha$$

$$\Delta S = Q(\Sigma_2) - Q(\Sigma_1) + \int_R d^4x\, \mathcal{E}^\alpha \delta\phi^\alpha$$

where the surface integral $\quad Q(\Sigma_i) = \int_{\Sigma_i} d\sigma_\mu \left(\mathcal{L}\,\delta^\mu_\nu \delta x^\nu - \pi^\mu_\alpha \partial_\nu \phi^\alpha \delta x^\nu + \pi^\mu_\alpha \delta'\phi^\alpha \right)$

can be expressed in terms of canonical stress-energy tensor

$$\Theta^\mu_\nu = \pi^\mu_\alpha \partial_\nu \phi^\alpha - \delta^\mu_\nu \mathcal{L}$$

as

$$Q(\Sigma_i) = \int_{\Sigma_i} d\sigma_\mu \left(\pi^\mu_\alpha \delta'\phi^\alpha - \Theta^\mu_\nu \delta x^\nu \right)$$

Now if the group of transformations introduced above is a symmetry group i.e. $\Delta S = 0$ and $\varepsilon^{\alpha} = 0$ we have

$$Q_k = \int_{\Sigma} d\sigma_{\mu} \left(\pi^{\mu}_{\alpha} \, G^{\alpha}_{k}(x,\phi) - \theta^{\mu}_{\nu} \, \Gamma^{\nu}_{k} \right) = \text{constant for any spacelike surface } \Sigma$$

i.e. Noether first theorem: invariance of the action $\Delta S = 0$ under a continuous p-parameter group of transformations $\delta x^{\mu} = \Gamma^{\mu}_{k}(x)\,\varepsilon^{k}$; $\delta' \phi^{\alpha} = G^{\alpha}_{k}(x,\phi)\,\varepsilon^{k}$ implies p global conservation laws for the integral quantities Q_k

With the use of Gauss theorem the global conservation laws are equivalent with a set of p equations of continuity

$$\partial_{\mu} \mathcal{N}^{\mu}_{k} = 0$$

with $\mathcal{N}^{\mu}_{k}$ → Noether currents

$$\mathcal{N}^{\mu}_{k} = \pi^{\mu}_{\alpha} \, G^{\alpha}_{k}(x,\phi) - \theta^{\mu}_{\nu} \, \Gamma^{\nu}_{k}(x)$$

$$\mathcal{N}^{\mu}_{k} = G^{\alpha}_{k}(x,\phi) \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi^{\alpha})} - \left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi^{\alpha})} \partial_{\nu}\phi^{\alpha} - \delta^{\mu}_{\nu} \mathcal{L} \right) \Gamma^{\nu}_{k}(x)$$

Noether currents associated to space-time translations (7N)

$x'^{\mu} = x^{\mu} + \epsilon^{\mu}$ $\quad \mu = 0,1,2,3 \Rightarrow$ four parameters $\Rightarrow \delta x^{\mu} = \epsilon^{\mu} = \delta^{\mu}{}_{\nu}\epsilon^{\nu}$

$\Gamma^{\mu}{}_{\alpha} = \dfrac{\partial \delta x^{\mu}}{\partial \epsilon^{\alpha}} \Rightarrow \Gamma^{\mu}{}_{\nu} = \dfrac{\partial(\delta x^{\mu})}{\partial \epsilon^{\nu}} = \delta^{\mu}{}_{\nu}$

Since at translations $\phi'^{\alpha}(x') = \phi^{\alpha}(x)$ i.e. $\delta'\phi^{\alpha} = 0 \Rightarrow G^{\alpha}_{\mu}(x,\phi) = 0$

Therefore the Noether currents (four Noether currents) are

$$W^{\mu}_{\nu}(x) = 0 - \Theta^{\mu}{}_{\alpha}\Gamma^{\alpha}{}_{\nu} = -\Theta^{\mu}{}_{\nu}$$

$$\Theta^{\mu\nu} = \pi^{\mu}_{\alpha}\partial^{\nu}\phi^{\alpha} - g^{\mu\nu}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi^{\alpha})}\partial^{\nu}\phi^{\alpha} - g^{\mu\nu}\mathcal{L}$$

Consequently, the Noether current associated with the translational invariance is identical to the canonical stress-energy tensor, and the local form of the conservation law (continuity equation) is:

$$\frac{\partial \Theta^{\mu\nu}}{\partial x^{\nu}} = 0$$

The corresponding conserved integral quantities are four-momenta of the fields

$$P^{\mu} = \int_{\Sigma} d\sigma_{\nu}\, \Theta^{\mu\nu}$$

or $\quad P^{\mu} = \int d^3x\, \Theta^{0\mu}$

$$\begin{cases} H = \int d^3x\, \Theta^{00} & \rightarrow \text{Total energy} \\ P^{i} = \int d^3x\, \Theta^{0i} & \rightarrow \text{Components of linear momentum} \end{cases}$$

Noether currents associated to Lorentz transformations (8N)

For $x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}$ with $\Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\ \nu} + \omega^{\mu}_{\ \nu} \Rightarrow \delta x^{\mu} = \omega^{\mu}_{\ \nu} x^{\nu}$;

Observe that:

$$\delta x^{\mu} = \omega^{\mu}_{\ \nu} x^{\nu} = \sum_{\rho<\sigma} \omega^{\rho\sigma} \left(\delta^{\mu}_{\rho} x_{\sigma} - \delta^{\mu}_{\sigma} x_{\rho} \right) = \sum_{\rho<\sigma} \left(\omega^{\mu\sigma} x_{\sigma} - \omega^{\rho\mu} x_{\rho} \right)$$

Then

$$\Gamma^{\mu}_{(\rho\sigma)} = \frac{\partial(\delta x^{\mu})}{\partial \omega^{\rho\sigma}} = \delta^{\mu}_{\rho} x_{\sigma} - \delta^{\mu}_{\sigma} x_{\rho} \Rightarrow \Gamma^{\mu(\rho\sigma)} = g^{\mu\rho} x^{\sigma} - g^{\mu\sigma} x^{\rho}$$

where $(\rho\sigma)$ – is an index for the parameter → six parameters

The corresponding total field variations are

$$\phi'(x') = e^{-\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}} \phi(x) \Rightarrow \delta'\phi(x) = \phi'(x') - \phi(x) = -\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu} \phi(x) \text{ or}$$

on field components:

$$\delta'\phi^{\alpha} = -\frac{i}{2}\omega_{\mu\nu} (J^{\mu\nu})^{\alpha}_{\ \beta} \phi^{\beta}.$$

Comments: for a scalar field $(J^{\mu\nu})^{\alpha}_{\ \beta} \equiv 0$

for a vector field $(J^{\mu\nu})^{\alpha}_{\ \beta} = i\left(g^{\mu\alpha}\delta^{\nu}_{\beta} - g^{\nu\alpha}\delta^{\mu}_{\beta}\right)$

for Dirac field $(J^{\mu\nu})^{\alpha}_{\ \beta} = \left(\frac{i}{4}[\gamma^{\mu},\gamma^{\nu}]\right)^{\alpha}_{\ \beta}$

Then

$$G^{(\rho\sigma)}_{\alpha} = \frac{\partial(\delta'\phi^{\alpha})}{\partial\omega_{\rho\sigma}} = -i\left(J^{\rho\sigma}\right)^{\alpha}_{\ \beta}\phi^{\beta} \equiv \left(I^{(\rho\sigma)}\right)^{\alpha}_{\ \beta}\phi^{\beta} \text{ so}$$

$$\left(I^{(\rho\sigma)}\right)^{\alpha}_{\ \beta} = -i\left(J^{\rho\sigma}\right)^{\alpha}_{\ \beta}$$

In this case the k parameter is the pair $(\rho\sigma)$, i.e. six indep. parameters

The Noether currents are: (angular momentum density)

$$\mathcal{N}^{\mu(\rho\sigma)}(x) \equiv \mathcal{M}^{\mu(\rho\sigma)}(x) = \pi^{\mu}_{\alpha} G^{(\rho\sigma)}_{\alpha}(x,\phi) - \Theta^{\mu}_{\ \nu}\Gamma^{\nu(\rho\sigma)} = \pi^{\mu}_{\alpha}\left(I^{(\rho\sigma)}\right)^{\alpha}_{\ \beta}\phi^{\beta} - \Theta^{\mu}_{\ \nu}\left(g^{\nu\rho}x^{\sigma} - g^{\nu\sigma}x^{\rho}\right) =$$

$$= \pi^{\mu}_{\alpha}\left(I^{(\rho\sigma)}\right)^{\alpha}_{\ \beta}\phi^{\beta} - \left(\Theta^{\mu\rho}x^{\sigma} - \Theta^{\mu\sigma}x^{\rho}\right) = \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi^{\alpha})}\left(I^{(\rho\sigma)}\right)^{\alpha}_{\ \beta}\phi^{\beta} + \Theta^{\mu\sigma}x^{\rho} - \Theta^{\mu\rho}x^{\sigma}$$

The global conserved quantity is an antisymmetric tensor of rank two: (ang. mom. tens)

$$M^{\rho\sigma} = \int_{\Sigma} d\sigma_{\mu}\mathcal{M}^{\mu(\rho\sigma)}(x) = \int d^3x \left(\Theta^{0\sigma}x^{\rho} - \Theta^{0\rho}x^{\sigma} + \frac{\partial\mathcal{L}}{\partial(\partial_0\phi^{\alpha})}\left(I^{(\rho\sigma)}\right)^{\alpha}_{\ \beta}\phi^{\beta}\right) = \underbrace{L^{\rho\sigma}}_{\text{orbital}} + \underbrace{S^{\rho\sigma}}_{\text{spin}}$$

angular momenta

The space components of $M^{\rho\sigma}$ give the familiar angular Momentum components

$$M^{ij} = \int d^3x \left(\Theta^{0i}x^{j} - \Theta^{0j}x^{i} + \frac{\partial\mathcal{L}}{\partial(\partial_0\phi^{\alpha})}\left(I^{(ij)}\right)^{\alpha}_{\ \beta}\phi^{\beta}\right)$$

Noether currents associated to an internal symmetry

The transformation by definition do not involve the spacetime coordinates transformation, i.e. $\delta x^\mu = 0$ and $\Gamma_k^\mu(x) = 0$. We have only $\delta' \phi^\alpha = G_k^\alpha(\phi)\epsilon^k$

Then the Noether current is $W_k^\mu = \pi_\alpha^\mu G_k^\alpha(\phi) = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)} G_k^\alpha(\phi)$

Let us consider the case where $\mathcal{L}$ corresponds to a description of complex fields ϕ^α and $\phi^{\alpha *}$. Then at a gauge (phase) transformation

$$\phi'^\alpha = e^{i\alpha}\phi^\alpha \Rightarrow \delta\phi'^\alpha = \phi'^\alpha - \phi^\alpha = i\alpha\phi^\alpha \Rightarrow G^\alpha(\phi) = i\phi^\alpha$$

$$\phi'^{\alpha *} = e^{-i\alpha}\phi^{\alpha *} \Rightarrow \delta\phi'^{\alpha *} = \phi'^{\alpha *} - \phi^{\alpha *} = -i\alpha\phi^{\alpha *} \Rightarrow G_*^\alpha(\phi) = -i\phi^{\alpha *}$$

and

$$j^\mu(x) = i\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)}\phi^\alpha - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{\alpha *})}\phi^{\alpha *}\right)$$

The global quantity correspond to the total charge

$$Q = i\int d^3x\, j^0(x) = i\int d^3x \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^\alpha)}\phi^\alpha - \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^{\alpha *})}\phi^{\alpha *}\right)$$

② DERIVATA FUNCTIONALĂ

- reprezintă diferențierea unei funcționale în raport cu argumentul său.

$F[\phi]$ – funcțională – aplicație de la spațiul linear al funcțiilor, $M = \{\phi(x) : x \in \mathbb{R}\}$ la mulțimea numerelor reale sau complexe

$F : M \to \mathbb{R}$ (sau $\mathbb{C}$)

Fie obiectul

$$\boxed{\frac{\delta F[\phi]}{\delta \phi(x)}}$$ → ne spune cum se schimbă valoarea funcționalei dacă funcția $\phi(x)$ se schimbă în punctul x

$$\delta F[\phi] = \int dx \, \frac{\delta F[\phi]}{\delta \phi(x)} \, \delta\phi(x)$$ → deci schimbarea totală a lui F la variația lui $\phi(x)$ este o superpoziție lineară a schimbărilor locale, sumat peste întreg domeniul de valori a lui x

Obs: Ca și în diferențierea ordinară, derivata funcțională poate fi reprezentată ca limita unei diferențe împărțită la variația corespunzătoare

Fie $\delta\phi(x) = \varepsilon\,\delta(x-y)$

Din (1) $\Rightarrow \delta F[\phi] = F[\phi + \varepsilon\delta(x-y)] - F[\phi] = \int dx \, \frac{\delta F[\phi]}{\delta\phi(x)} \varepsilon\delta(x-y) =$

$$= \varepsilon \frac{\delta F}{\delta \phi(y)}$$

$$\to \frac{\delta F[\phi]}{\delta \phi(y)} = \lim_{\varepsilon \to 0} \frac{F[\phi + \varepsilon\delta(x-y)] - F[\phi]}{\varepsilon}$$

- trebuie specificată ordinea operațiilor matematice

① prima trebuie luată limita $\varepsilon \to 0$

Obs: Derivata funcțională este o operație lineară. Multe reguli ale calculului diferențial se pot extrapola la derivata funcțională

- regula produsului : fie $F[\phi] = G[\phi] H[\phi]$

$$\frac{\delta F[\phi]}{\delta \phi(x)} = \frac{\delta G[\phi]}{\delta \phi(x)} H[\phi] + G[\phi] \frac{\delta H[\phi]}{\delta \phi(x)}$$

- „chain rule"

$$\frac{\delta}{\delta\phi(y)} F[G[\phi]] = \int dx \, \frac{\delta F[G]}{\delta G(x)} \, \frac{\delta G[\phi]}{\delta \phi(y)}$$

Exemple utile

1. $$\boxed{F[\phi] = \int dx\,(\phi(x))^n}$$

Din (4) $\Rightarrow$

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx\,(\phi(x)+\epsilon\delta(x-y))^n - \int dx\,(\phi(x))^n\right)$$

$$= \int dx\, n(\phi(x))^{n-1}\delta(x-y) = n(\phi(y))^{n-1}$$

$\delta F[\phi] = \int dy\, n(\phi(y))^{n-1}\delta\phi(y)$

Obs: in acest caz derivarea funcțională anulează integrarea. Limita $\underline{\epsilon\to 0}$ este luată înaintea integrării / altfel expresii nedefinite de forma $(\delta(x-y))^2$ vor apărea

O generalizare imediată la funcții $g(\phi(x))$ în integrand este

$$\frac{\delta}{\delta\phi(y)}\int dx\, g(\phi(x)) = g'(\phi(y))$$

2. $$\boxed{F[\phi] = \int dx\left(\frac{d\phi(x)}{dx}\right)^n}$$

(4) $\Rightarrow$ $$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx\left(\frac{d}{dx}[\phi(x)+\epsilon\delta(x-y)]\right)^n - \int dx\left(\frac{d}{dx}\phi(x)\right)^n\right) =$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int dx\left(\cancel{\left(\frac{d\phi}{dx}\right)^n} + n\epsilon\left(\frac{d\phi}{dx}\right)^{n-1}\frac{d}{dx}\delta(x-y) + O(\epsilon^2) - \cancel{\left(\frac{d\phi}{dx}\right)^n}\right) =$$

$$= n\int dx\left(\frac{d\phi}{dx}\right)^{n-1}\frac{d}{dx}\delta(x-y) = -n\frac{d}{dx}\left(\frac{d\phi}{dx}\right)^{n-1}\Big|_y$$

Similar pentru o funcție $h(d\phi/dx)$ avem

$$\frac{\delta}{\delta\phi(y)}\int dx\, h\left(\frac{d\phi}{dx}\right) = -\frac{d}{dx}\frac{dh}{d\left(\frac{d\phi}{dx}\right)}\Big|_y$$

3. $$\boxed{F[\phi] = \int dx'\, K(y,x')\phi(x')}$$

$$\frac{\delta F_y[\phi]}{\delta\phi(x)} = K(y,x)$$

) LEGI DE CONSERVARE ÎN TEORIILE CLASICE DE CÎMP

TEOREMA NOETHER

„La fiecare transformare continuă de simetrie corespunde o lege de conservare

* Fie

$$x'_\mu = x_\mu + \delta x_\mu$$

Atunci modificarea corespunzătoare a cîmpului este

$$\phi'_r(x') = \phi_r(x) + \delta\phi_r(x)$$

iar densitatea de Lagrangean trece în

$$\mathcal{L}'(x') = \mathcal{L}(x) + \delta\mathcal{L}(x)$$

$\left(\text{Obs: } \mathcal{L}(x) = \mathcal{L}\left(\phi(x), \frac{\partial\phi}{\partial x_\mu}\right)\right)$

$$\tilde{\delta}\phi_r(x) = \phi'_r(x) - \phi_r(x)$$

Obs: Cele două tipuri de variații sunt legate prin:

$$\tilde{\delta}\phi_r(x) = \phi'_r(x) - \phi'_r(x') + \phi'_r(x') - \phi_r(x) =$$

$$= \delta\phi_r(x) - \left(\phi'_r(x') - \phi'_r(x)\right) = \delta\phi_r(x) - \frac{\partial\phi'_r(x)}{\partial x_\mu}\delta x_\mu$$

deci

$$\tilde{\delta}\phi_r(x) = \delta\phi_r(x) - \frac{\partial\phi_r}{\partial x_\mu}\delta x_\mu$$

Obs. Variația modificată $\tilde{\delta}$ comută cu diferențierea $\frac{\partial}{\partial x_\mu}$

$$\frac{\partial}{\partial x_\mu}\tilde{\delta}\phi_r(x) = \tilde{\delta}\left(\frac{\partial\phi_r(x)}{\partial x_\mu}\right)$$

Obs: Variația δ, în schimb, nu comută cu operația de diferențiere

$$\frac{\partial}{\partial x_\mu}\left(\delta\phi_r(x)\right) = \frac{\partial}{\partial x_\mu}\phi'_r(x') - \frac{\partial}{\partial x_\mu}\phi_r(x) =$$

$$= \left(\frac{\partial\phi'_r(x')}{\partial x'_\mu} - \frac{\partial\phi_r(x)}{\partial x_\mu}\right) + \frac{\partial\phi'_r(x')}{\partial x_\mu} - \frac{\partial\phi'_r(x')}{\partial x'_\mu}$$

$$= \delta\left(\frac{\partial\phi_r(x)}{\partial x_\mu}\right) + \frac{\partial x'^\nu}{\partial x_\mu}\frac{\partial\phi'_r(x')}{\partial x'^\nu} - \frac{\partial\phi'_r(x')}{\partial x'_\mu}$$

dar $\dfrac{\partial x'^\nu}{\partial x_\mu} = g^{\nu\mu} + \dfrac{\partial\delta x^\nu}{\partial x_\mu}$

$$= \delta\left(\frac{\partial\phi_r(x)}{\partial x_\mu}\right) + \frac{\partial\phi'_r(x')}{\partial x'^\nu}\frac{\partial\delta x^\nu}{\partial x_\mu}$$

$$= \delta\left(\frac{\partial\phi_r(x)}{\partial x_\mu}\right) + \frac{\partial\phi_r(x)}{\partial x^\nu}\frac{\partial\delta x^\nu}{\partial x_\mu}$$

* Presupunem că transformarea infinitezimală lasă acțiunea integrală invariantă:

$$\delta W \equiv \int_{\Omega'} d^4x'\,\mathcal{L}'(x') - \int_\Omega d^4x\,\mathcal{L}(x) = 0$$

Ω' reprezintă același volum de integrare ca și Ω exprimat în noile coordonate

Dar

$$\delta W = \int_{\Omega'} d^4x'\,\delta\mathcal{L}(x) + \int_{\Omega'} d^4x'\,\mathcal{L}(x) - \int_\Omega d^4x\,\mathcal{L}(x)$$

$$d^4x' = \left|\frac{\partial(x'^\mu)}{\partial(x^\nu)}\right| d^4x = \begin{vmatrix} 1+\frac{\partial\delta x_0}{\partial x_0} & \frac{\partial\delta x_0}{\partial x_1} & \cdots & \cdots \\ \frac{\partial\delta x_1}{\partial x_0} & 1+\frac{\partial\delta x_1}{\partial x_1} & \cdots & \cdots \\ \vdots & \vdots & \vdots & 1+\frac{\partial\delta x_3}{\ } \end{vmatrix} d^4x =$$

With $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ we proved that

$$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} + \eta^{\mu\sigma} J^{\nu\rho} - \eta^{\nu\sigma} J^{\mu\rho}$$

With $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) = -\eta^{\mu\nu}$ multiply by i both sides of the commutation relation:

Then:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left[g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right]$$ (as in Michele Maggiore pag 19)

Remember that $J^{\mu\nu} = -J^{\nu\mu}$

If $J^i = \frac{1}{2}\varepsilon^{ijk} J^{jk}$ $\Rightarrow$ $\varepsilon^{lmi} J^i = \frac{1}{2}\varepsilon^{lmi}\varepsilon^{jki} J^{jk} = \frac{1}{2}(\delta^{lj}\delta^{mk} - \delta^{lk}\delta^{mj}) J^{jk} =$ (Peskin pag 39)
$= J^{lm}$

$K^i = J^{i0}$

then the Lie algebra of the Lorentz group becomes

$$[J^i, J^j] = i\varepsilon^{ijk} J^k$$

$$[J^i, K^j] = i\varepsilon^{ijk} K^k$$

$$[K^i, K^j] = -i\varepsilon^{ijk} J^k$$

Proof: $[J^i, J^j] = \frac{1}{4}\varepsilon^{ilm}\varepsilon^{jpq}[J^{lm}, J^{pq}] = \frac{i}{4}\varepsilon^{ilm}\varepsilon^{jpq}\left(g^{mp} J^{lq} - g^{lp} J^{mq} - g^{mq} J^{lp} + g^{lq} J^{mp}\right)$

$= -\frac{i}{4}\varepsilon^{ilm}\varepsilon^{jpq}\left(\delta^{mp} J^{lq} - \delta^{lp} J^{mq} - \delta^{mq} J^{lp} + \delta^{lq} J^{mp}\right) =$

$= -\frac{i}{4}\varepsilon^{ilm}\varepsilon^{jmq}\varepsilon^{lqk} J^k \rightarrow +\frac{i}{4}(\delta^{ij}\delta^{lq} - \delta^{iq}\delta^{lj})\varepsilon^{lqk} J^k = -\frac{i}{4}\varepsilon^{jik} J^k = \frac{i}{4}\varepsilon^{ijk} J^k$

$+\frac{i}{4}\varepsilon^{ilm}\varepsilon^{jlq}\varepsilon^{mqk} J^k \rightarrow \frac{i}{4}(\delta^{ij}\delta^{mq} - \delta^{iq}\delta^{mj})\varepsilon^{mqk} J^k = -\frac{i}{4}\varepsilon^{jik} J^k = \frac{i}{4}\varepsilon^{ijk} J^k$

$+\frac{i}{4}\varepsilon^{ilm}\varepsilon^{jpm}\varepsilon^{lpk} J^k$

$-\frac{i}{4}\varepsilon^{ilm}\varepsilon^{jpl}\varepsilon^{mpk} J^k$

$= i\varepsilon^{ijk} J^k$ q.e.d.

$[J^i, K^j] = \frac{1}{2}\varepsilon^{ilm}[J^{lm}, J^{j0}] = \frac{i}{2}\varepsilon^{ilm}\left(g^{l0} J^{mj} + g^{mj} J^{l0} - g^{lj} J^{m0} - g^{m0} J^{lj}\right) =$

$= \frac{i}{2}\varepsilon^{ilm}(-)\delta^{mj} K^l - \frac{i}{2}\varepsilon^{ilm}(-)\delta^{lj} K^m \overset{!}{=} -\frac{i}{2}\varepsilon^{ilj} K^l + \frac{i}{2}\varepsilon^{ijm} K^m = +\frac{i}{2}\varepsilon^{ijk} K^k + \frac{i}{2}\varepsilon^{ijk} K^k$

$= +i\varepsilon^{ijk} K^k$ q.e.d.

$[K^i, K^j] = [J^{i0}, J^{j0}] = i\left(\delta^{i0} J^{0j} + \delta^{0j}\delta^{i0} - \delta^{ij} J^{00} - \delta^{00} J^{ij}\right) = -i\varepsilon^{ijk} J^k$ q.e.d.

deci în ordinul întîi:

$$d^4x' = \left(1+\frac{\partial \delta x^\mu}{\partial x^\mu}\right)d^4x$$

și variația acțiunii integrale devine (în ordinul întîi)

$$\delta W = \int_\Omega d^4x\, \delta\mathcal{L}(x) + \int_\Omega d^4x\, \frac{\partial \delta x^\mu}{\partial x^\mu}\mathcal{L}(x)$$

$$= \int_\Omega d^4x\left(\tilde{\delta}\mathcal{L}(x) + \frac{\partial\mathcal{L}(x)}{\partial x^\mu}\delta x^\mu\right) + \int_\Omega d^4x\, \mathcal{L}(x)\frac{\partial\delta x^\mu}{\partial x^\mu}$$

În continuare exprimăm variația totală $\tilde{\delta}\mathcal{L}(x)$ în termeni de variația cîmpurilor și a derivatelor cîmpurilor

$$\tilde{\delta}\mathcal{L}(x) = \frac{\partial\mathcal{L}(x)}{\partial\phi_r}\tilde{\delta}\phi_r(x) + \frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\tilde{\delta}\left(\frac{\partial\phi_r(x)}{\partial x^\mu}\right)$$

$$= \frac{\partial\mathcal{L}(x)}{\partial\phi_r}\tilde{\delta}\phi_r(x) - \frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\right)\tilde{\delta}\phi_r(x)$$

$$+\frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\right)\tilde{\delta}\phi_r(x) + \frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\frac{\partial}{\partial x_\mu}\left(\tilde{\delta}\phi_r(x)\right)$$

$$= \left[\frac{\partial\mathcal{L}(x)}{\partial\phi_r} - \frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\right)\right]\tilde{\delta}\phi_r(x)$$

$$+\frac{\partial}{\partial x_\mu}\left[\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\tilde{\delta}\phi_r(x)\right]$$

sumare după indicii Minkowski — sumare după r

Acum

$$0 = \delta W = \int_\Omega d^4x\,\underbrace{\left(\tilde{\delta}\mathcal{L}(x) + \frac{\partial}{\partial x^\mu}\left(\mathcal{L}(x)\,\delta x^\mu\right)\right)}$$

Ω arbitrar $\Rightarrow$ integrandul trebuie să se anuleze

$$=\left[\frac{\partial\mathcal{L}(x)}{\partial\phi_r} - \frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\right)\right]\tilde{\delta}\phi_r(x) + \frac{\partial}{\partial x^\mu}\left[\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi_r)}\tilde{\delta}\phi_r(x) + \mathcal{L}(x)\delta x^\mu\right]$$

$\Rightarrow$ O ecuație de continuitate

$$\frac{\partial}{\partial x_\mu} f^\mu(x) = 0$$

pentru densitatea de curent.

$$f_\mu(x) = \frac{\partial \mathcal{L}(x)}{\partial(\partial^\mu \phi_r)} \delta\phi_r(x) - \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_r)} \frac{\partial \phi_r}{\partial x^\nu} - g_{\mu\nu} \mathcal{L}(x)\right) \delta x^\nu$$

unde am ~~[illegible]~~ exprimat $\tilde{\delta}\phi_r(x)$ în termeni de $\delta\phi_r$

ce reprezintă legea de conservare în formă diferențială

Obs: Cantitatea conservată se poate obține integrând peste volumul spațial tridimensional:

$$0 = \int_V d^3x \frac{\partial}{\partial x_\mu} f_\mu(x) = \int_V d^3x \frac{\partial}{\partial x_0} f_0(x) + \int_V d^3x \, \vec{\nabla} \cdot \vec{f}(x) =$$

$$= \frac{d}{dx_0} \int_V d^3x \, f_0(x) + \underbrace{\oint_{\partial V} d\sigma \cdot f(x)}_{0}$$

Deci

$$G := \int_V d^3x \, f_0(x)$$

reprezintă ~~contit~~ mărimea conservată, având o valoare constantă în timp a] Invarianța la translații:

Fie transformarea de coordonate

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}$$

Omogenitatea spațiu-timpului $\Rightarrow$ acțiunea integrală invariantă

Acum $\phi'_r(x') = \phi_r(x)$ → forma câmpului nu se modifică la translație

$$\Rightarrow \delta\phi_r = 0$$

cf Noether $\Rightarrow$

$$\frac{\partial}{\partial x_{\mu}} \Theta_{\mu\nu} = 0$$

unde

$$\Theta_{\mu\nu} = -g_{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi_r)}\frac{\partial\phi_r}{\partial x^{\nu}}$$ → Tensorul canonic energie-impuls

$\nu = 0, 1, 2, 3 \Rightarrow$ 4 mărimi conservate

E – energia

$\vec{P}$ – vectorul impuls

$$P^{\nu} = \left(\frac{E}{c}, \vec{P}\right) = \frac{1}{c}\int_V d^3x\, \Theta^{0\nu}(x) = \text{Constant}$$

$\boxed{b}$ Invarianța la transformările Lorentz

Fie o rotație infinitesimală generală (în spațiul Minkowski)

$$x'^{\mu} = x^{\mu} + \delta\omega^{\mu\nu} x_{\nu}$$

$$\delta\omega^{\mu\nu} = -\delta\omega^{\nu\mu} \rightarrow \text{depinde de unghiurile de rotație (în patru dimen}$$

• Conservarea lungimii vectorului x^{μ} $\xRightarrow{\text{Demons}}$ $x'^{\mu} x'_{\mu} = (x^{\mu} + \delta\omega^{\mu\nu} x_{\nu})(x_{\mu} + \delta\omega_{\mu}{}^{\tau} x_{\tau})$

$$= x^{\mu} x_{\mu} + x_{\mu} x_{\nu} \delta\omega^{\mu\nu} + \delta\omega_{\mu}{}^{\tau} x^{\mu} x_{\tau}$$

Spațiul isotropic + princ. relativit. $\Rightarrow$ invarianța acțiunii integrale la transf. Lorentz

Transformarea funcțiilor câmpului

$$\phi'_r(x') = \phi_r(x) + \frac{1}{2}\delta\omega_{\mu\nu}\left(I^{\mu\nu}\right)_{rs}\phi_s(x)$$

Câmpurile fizice (clasice) ϕ_r se transformă cu ajutorul unei reprezentări ireductibile a grupului Lorentz

$I^{\mu\nu}$ → generatorii infinitezimali ai transformării Lorentz → 6 generatori independenți — 3 rotații, 3 boosts

$(I^{\mu\nu})_{rs}$ → elementul de matrice al reprezentării generatorului infinitezimal corespunzător

Atunci „densitatea" conservată este

$$f_{\mu}(x) = \frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi_r)}\frac{1}{2}\delta\omega_{\nu\lambda}\left(I^{\nu\lambda}\right)_{rs}\phi_s(x) - \Theta_{\mu\nu}\,\delta\omega^{\nu\lambda}x_{\lambda}$$

cu $\Theta_{\mu\nu}$ → tensorul energie-impuls

sau cu $\Theta_{\mu\nu}\delta\omega^{\nu\lambda}x_{\lambda} = \frac{1}{2}\delta\omega^{\nu\lambda}\left(\Theta_{\mu\nu}x_{\lambda} - \Theta_{\mu\lambda}x_{\nu}\right)$

$$f_{\mu}(x) = \frac{1}{2}\delta\omega^{\nu\lambda} M_{\mu\nu\lambda}(x)$$

unde am folosit notația

$$M_{\mu\nu\lambda}(x) = \Theta_{\mu\lambda} x_\nu - \Theta_{\mu\nu} x_\lambda + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_r)} (I_{\nu\lambda})_{rs} \phi_s(x)$$

Mărimea globală care se conservă este un tensor antisimetric

$$M_{\nu\lambda} = \int d^3x \left[\Theta_{0\lambda} x_\nu - \Theta_{0\nu} x_\lambda + \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_r)} (I_{\nu\lambda})_{rs} \phi_s(x) \right]$$

componentele spațiale → sunt legate de momentul cinetic orbital și de spin (intern)

$$M_{nl} = L_{nl} + S_{nl}$$

unde

descrie momentul cinetic orbital

$$L_{nl} = \int d^3x \left(x_n \Theta_{0l} - x_l \Theta_{0n} \right) =$$

$$= \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_r)} \left(x_n \frac{\partial}{\partial x^l} - x_l \frac{\partial}{\partial x^n} \right) \phi_r(x)$$

descrie momentul cinetic intern (de spin)

$$S_{nl} = \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_r)} \underline{\left(I^{nl} \right)_{rs}} \phi_s(x)$$

componentele mixte (spațiu-timp) → alte trei mărimi conservate legate de generalizarea relativistă a centrului de masă

Observe that with $\omega^i = \frac{1}{2}\varepsilon^{ijk}\omega^{jk}$; $\nu^i = \omega^{i0}$

$J^i = \frac{1}{2}\varepsilon^{ijk}J^{jk}$; $K^i = J^{i0}$

(28)

$$\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} = \frac{1}{2}g_{\mu\alpha}g_{\nu\beta}\omega^{\alpha\beta}J^{\mu\nu} = \frac{1}{2}g_{\mu l}g_{\nu m}\omega^{lm}J^{\mu\nu} + \frac{1}{2}g_{\mu 0}g_{\nu i}\omega^{0i}J^{\mu\nu} + \frac{1}{2}g_{\mu i}g_{\nu 0}\omega^{i0}J^{\mu\nu}$$

$$= \frac{1}{2}g_{\mu l}g_{\nu m}\omega^{lm}J^{\mu\nu} + \frac{1}{2}\omega^{0i}g_{\mu 0}g_{\nu i}J^{\mu\nu} + \frac{1}{2}\underbrace{g_{\mu i}g_{\nu 0}}_{(-)}\omega^{i0}J^{\mu\nu}$$

$$= \frac{1}{2}\omega_{ij}J^{ij} + \frac{1}{2}\omega_{i0}J^{i0} + \frac{1}{2}\omega_{0i}J^{0i} =$$

$$= \frac{1}{2}\varepsilon^{lmk}\omega^k J^{lm} + (-)\omega^{i0}J^{i0} = \omega^k J^k - \nu^i K^i = \vec{\omega}\cdot\vec{J} - \vec{\nu}\cdot\vec{K}$$

$$\boxed{\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} = \vec{\omega}\cdot\vec{J} - \vec{\nu}\cdot\vec{K}}$$

$$\omega^1 = \frac{1}{2}\varepsilon^{123}\omega^{23} - \frac{1}{2}\varepsilon^{132}\omega^{32} = \omega^{23} = \omega_{23}$$

$\omega_{ij} = \omega^{ij}$

$$\omega_{ij} = \underbrace{g_{il}g_{jm}}_{+1}\omega^{lm} \qquad \omega_{23} = \omega^{23}$$

$$\omega_{i0} = \underbrace{g_{i\mu}}_{+1}\underbrace{g_{0\nu}}_{-1}\omega^{\mu\nu} = -\omega^{i0} = -\nu^i$$

With $\vec{A} = \frac{1}{2}(\vec{J} + i\vec{K})$

$\vec{B} = \frac{1}{2}(\vec{J} - i\vec{K})$ $\Rightarrow$ $\left\{ \begin{array}{l} \vec{J} = \vec{A} + \vec{B} \\ \vec{K} = \dfrac{\vec{A} - \vec{B}}{i} \end{array} \right.$

Observations:

$$[A^i, B^j] = \frac{1}{4}[J^i + iK^i, J^j - iK^j] = \frac{1}{4}\left([J^i, J^j] - i[J^i, K^j] + i[K^i, J^j] + [K^i, K^j]\right)$$

$$= \frac{1}{4}\left(\cancel{i\varepsilon^{ijk}J^k} + (-)i\,\cancel{i\varepsilon^{ijk}K^k} + i\,\cancel{i\varepsilon^{jik}K^k} - \cancel{i\varepsilon^{ijk}J^k}\right) = 0$$

$$[A^i, A^j] = \frac{1}{4}[J^i + iK^i, J^j + iK^j] = \frac{1}{4}\left([J^i, J^j] + i[J^i, K^j] + i[K^i, J^j] - [K^i, K^j]\right)$$

$$= \frac{1}{4}\left(i\varepsilon^{ijk}J^k + i\,i\varepsilon^{ijk}K^k - i\,i\varepsilon^{jik}K^k - (-)i\varepsilon^{ijk}J^k\right) =$$

$$= \frac{1}{2}i\varepsilon^{ijk}(J^k + iK^k) = i\varepsilon^{ijk}A^k$$

$$[B^i, B^j] = i\varepsilon^{ijk}B^k$$

WEYL FIELDS

(j^B, j^A) → the representations of Lorentz algebra can be labeled by the two half-integers (j^B, j^A), the dimension of the representation being $(2j^B+1)(2j^A+1)$

The representation $(\frac{1}{2}, 0)$

$\vec{B} = \frac{\vec{\sigma}}{2}$; $\vec{A} = 0 \Rightarrow \begin{cases} \vec{J} = \frac{\vec{\sigma}}{2} \\ \vec{K} = i\frac{\vec{\sigma}}{2} \end{cases}$ $\qquad U_L(\vec{\omega}, \vec{\nu}) = e^{(-i\vec{\omega} - \vec{\nu})\cdot\frac{\vec{\sigma}}{2}}$

The representation $(0, \frac{1}{2})$

$\vec{B} = 0$; $\vec{A} = \frac{\vec{\sigma}}{2} \Rightarrow \begin{cases} \vec{J} = \frac{\vec{\sigma}}{2} \\ \vec{K} = -i\frac{\vec{\sigma}}{2} \end{cases}$ $\qquad U_R(\vec{\omega}, \vec{\nu}) = e^{(-i\vec{\omega} + \vec{\nu})\cdot\frac{\vec{\sigma}}{2}}$

Important observations (Here we use $\vec{\sigma}^* = -\sigma_2\vec{\sigma}\sigma_2$)

① The hermitian conjugate of U_L (U_R) is equal to the inverse of U_R (U_L)

Proof: $U_L^\dagger = \left(e^{(-i\vec{\omega} - \vec{\nu})\cdot\frac{\vec{\sigma}}{2}}\right)^\dagger = e^{(i\vec{\omega} - \vec{\nu})\frac{\vec{\sigma}}{2}} = e^{(-i(-\vec{\omega}) + (-\vec{\nu}))\frac{\vec{\sigma}}{2}} = U_R^{-1}$

$U_R^\dagger = \left(e^{(-i\vec{\omega} + \vec{\nu})\cdot\frac{\vec{\sigma}}{2}}\right)^\dagger = e^{(i\vec{\omega} + \vec{\nu})\frac{\vec{\sigma}}{2}} = e^{(-i(-\vec{\omega}) - (-\vec{\nu}))\frac{\vec{\sigma}}{2}} = U_L^{-1}$

② The representations are not real, in the sense that by complex conjugation we do not obtain a similarity transformation as in the case of rotations but

$U_L^* = \left(e^{(-i\vec{\omega} - \vec{\nu})\cdot\frac{\vec{\sigma}}{2}}\right)^* = e^{(i\vec{\omega} + \vec{\nu})\frac{\vec{\sigma}^*}{2}} = e^{(i\vec{\omega} - \vec{\nu})\frac{(-)\sigma_2\vec{\sigma}\sigma_2}{2}} = \sigma_2 U_R \sigma_2 = \sigma_2 U_R \sigma_2^{-1}$

$U_R^* = \left(e^{(-i\vec{\omega} + \vec{\nu})\cdot\frac{\vec{\sigma}}{2}}\right)^* = e^{(i\vec{\omega} + \vec{\nu})\frac{\vec{\sigma}^*}{2}} = e^{(i\vec{\omega} + \vec{\nu})\frac{(-)\sigma_2\vec{\sigma}\sigma_2}{2}} = \sigma_2 U_L \sigma_2 = \sigma_2 U_L \sigma_2^{-1}$,

③ Under a parity transformation $(t, \vec{x}) \rightarrow (t, -\vec{x})$, the boost generators behave as true vectors and change sign: $\vec{K} \rightarrow -\vec{K}$ since the parity transf reverses the velocity $\vec{v}$ of the boost

The angular momentum generators $\vec{J}$ are instead pseudovectors i.e $\vec{J} \rightarrow \vec{J}$

Therefore under parity $\vec{A} \xrightleftharpoons{\text{PARITY}} \vec{B}$ and consequently $U_L \xrightleftharpoons{\text{PARITY}} U_R$

4F

Spinors:

1. Left-handed (Weyl) spinor $\psi_L(x)$: a two-components object which transforms with the representation $U_L(\vec{\omega},\vec{\nu})$ at a Lorentz transformation characterized by parameters $\omega_{\mu\nu}$ i.e

$$\psi_L'(x') = U_L\, \psi_L(x) \qquad \text{under } x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

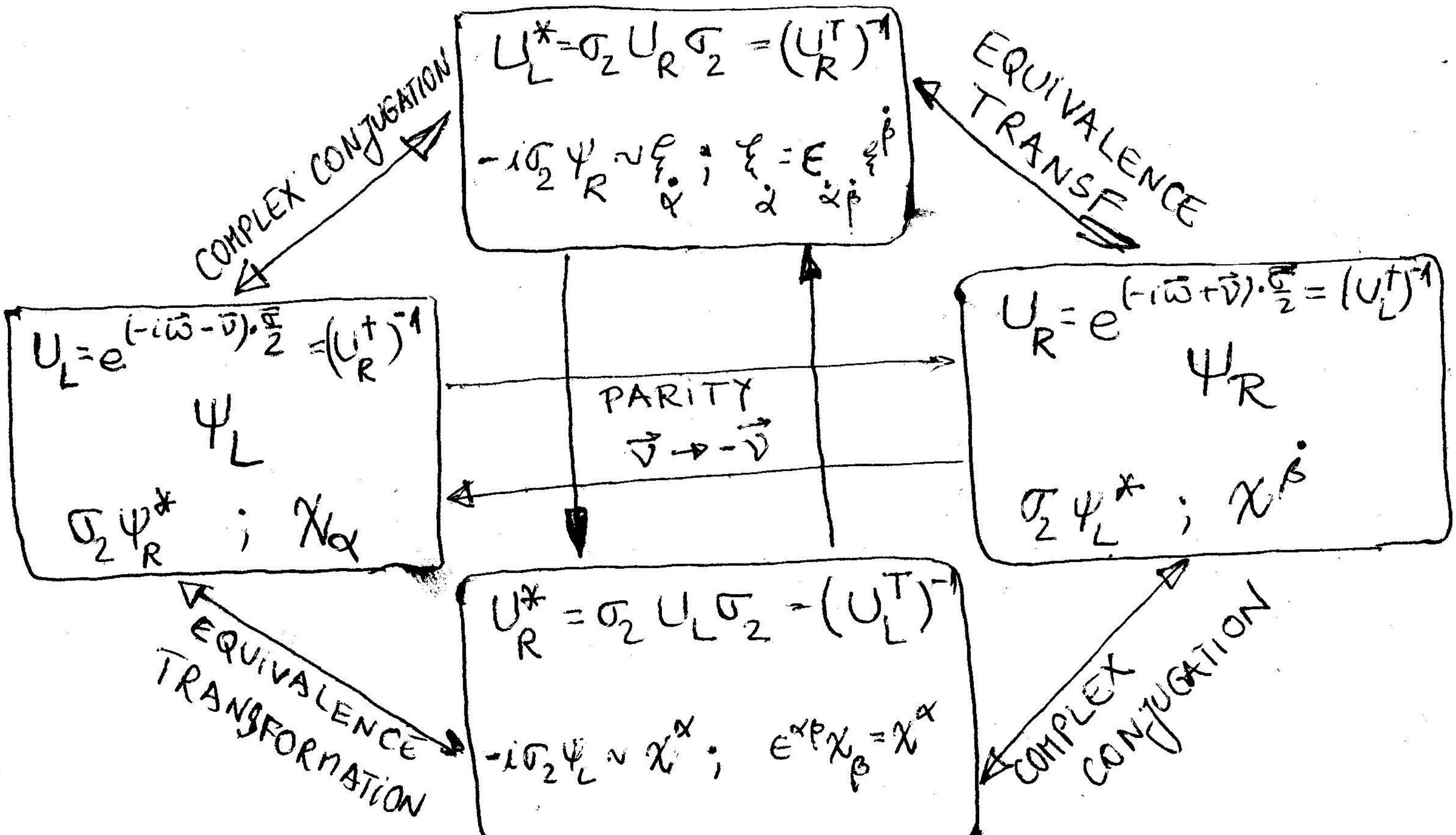
i.e transforms with $(\frac{1}{2}, 0)$ representation of Lorentz group, of dimension two

2. Right-handed Weyl spinor $\psi_R(x)$ is a two-components object which under the Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ transforms with the $(0, \frac{1}{2})$ representation i.e

$$\psi_R'(x') = U_R\, \psi_R(x)$$

3. The spinor $\sigma_2 \psi_R^*(x)$ transforms as $\psi_L(x)$ i.e. is a left-handed spinor
Indeed: $(\sigma_2 \psi_R^*(x'))' = \sigma_2 U_R^* \psi_R^*(x) = \underbrace{\sigma_2\sigma_2}_{I} U_L \sigma_2 \psi_R^*(x) = U_L\left(\sigma_2 \psi_R^*(x)\right)$

4. The spinor $\sigma_2 \psi_L^*(x)$ transforms as $\psi_R(x)$ i.e transforms with right representation.
Indeed $(\sigma_2 \psi_L^*(x'))' = \sigma_2 U_L^* \psi_L^*(x) = \sigma_2\sigma_2 U_R \sigma_2 \psi_L^* = U_R\left(\sigma_2 \psi_L^*(x)\right)$ q.e.d.



DIRAC FIELD

A field with four components $\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix}$ which at a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ transform as:

$$\psi'(x') = U_D \, \psi(x) = \begin{pmatrix} U_L & 0 \\ 0 & U_R \end{pmatrix} \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix}$$

Comment: FINITE-DIMENSIONAL REPRESENTATIONS OF LORENTZ GROUP

Method due to Dirac: For any set of four $n \times n$ matrices γ^μ satisfying the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2 g^{\mu\nu} I_{n\times n}$$

a n-dimensional representation of the Lorentz algebra is

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

Proof: Show directly from the definition that $S^{\mu\nu}$ verify the Lorentz algebra i.e. the commutation relations

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\left(g^{\nu\rho} S^{\mu\sigma} + g^{\mu\sigma} S^{\nu\rho} - g^{\mu\rho} S^{\nu\sigma} - g^{\nu\sigma} S^{\mu\rho}\right)$$

Indeed, since $[\gamma^\mu, \gamma^\nu] = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu = \{\gamma^\mu, \gamma^\nu\} - 2\gamma^\nu\gamma^\mu = 2g^{\mu\nu} I - 2\gamma^\nu\gamma^\mu$

$$[S^{\mu\nu}, S^{\rho\sigma}] = \left[\frac{i}{4}[\gamma^\mu,\gamma^\nu], \frac{i}{4}[\gamma^\rho,\gamma^\sigma]\right] = \frac{i^2}{16}\left[2g^{\mu\nu}I - 2\gamma^\nu\gamma^\mu, 2g^{\rho\sigma}I - 2\gamma^\sigma\gamma^\rho\right]$$

$$= \frac{i^2}{16} 4 \left[\gamma^\nu\gamma^\mu, \gamma^\sigma\gamma^\rho\right] = \frac{4}{16} i^2 \left(\gamma^\nu[\gamma^\mu, \gamma^\sigma\gamma^\rho] + [\gamma^\nu, \gamma^\sigma\gamma^\rho]\gamma^\mu\right) =$$

$$= \frac{1}{4} i^2 \left(\gamma^\nu\gamma^\sigma[\gamma^\mu,\gamma^\rho] + \gamma^\nu[\gamma^\mu,\gamma^\sigma]\gamma^\rho + \gamma^\sigma[\gamma^\nu,\gamma^\rho]\gamma^\mu + [\gamma^\nu,\gamma^\sigma]\gamma^\rho\gamma^\mu\right) =$$

$$= i\frac{i}{4}\Big(\cancel{\gamma^\nu\gamma^\sigma\gamma^\mu\gamma^\rho} - \gamma^\nu\gamma^\sigma\gamma^\rho\gamma^\mu + \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho - \cancel{\gamma^\nu\gamma^\sigma\gamma^\mu\gamma^\rho} + \gamma^\sigma\gamma^\nu\gamma^\rho\gamma^\mu - \gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu$$
$$+ \gamma^\nu\gamma^\sigma\gamma^\rho\gamma^\mu - \gamma^\sigma\gamma^\nu\gamma^\rho\gamma^\mu\Big) =$$

$$= i\frac{i}{4}\left(\cancel{-2g^{\nu\sigma}\gamma^\rho\gamma^\mu} + \cancel{2g^{\nu\sigma}\gamma^\rho\gamma^\mu} + \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho - \gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu\right)$$

$$= i\frac{i}{4}\left(\gamma^\nu(2g^{\mu\sigma} - \gamma^\sigma\gamma^\mu)\gamma^\rho - \gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu\right) = i\frac{i}{4}\left(2g^{\mu\sigma}\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\sigma\gamma^\mu\gamma^\rho - \gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu\right) = i\frac{i}{4}\left(2g^{\mu\sigma}\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\sigma(2g^{\mu\rho} - \gamma^\rho\gamma^\mu) - \gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu\right) =$$

$$= i\frac{i}{4}\left(2g^{\mu\sigma}\gamma^\nu\gamma^\rho - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + \gamma^\nu\gamma^\sigma\gamma^\rho\gamma^\mu - \gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu\right) =$$

$$= i\frac{i}{4}\left(2g^{\mu\sigma}\gamma^\nu\gamma^\rho - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + (2g^{\nu\sigma} - \gamma^\sigma\gamma^\nu)\gamma^\rho\gamma^\mu - \gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu\right) =$$

(6F)

$$= i\frac{i}{4}\left(2g^{\mu\sigma}\gamma^{\nu}\gamma^{\rho} - 2g^{\mu\rho}\gamma^{\nu}\gamma^{\sigma} + 2g^{\nu\sigma}\gamma^{\rho}\gamma^{\mu} - \gamma^{\sigma}\gamma^{\nu}\gamma^{\rho}\gamma^{\mu} - \gamma^{\sigma}\gamma^{\rho}\gamma^{\nu}\gamma^{\mu}\right) =$$

$$= i\frac{i}{4}\left(2g^{\mu\sigma}\gamma^{\nu}\gamma^{\rho} - 2g^{\mu\rho}\gamma^{\nu}\gamma^{\sigma} + 2g^{\nu\sigma}\gamma^{\rho}\gamma^{\mu} - 2g^{\nu\rho}\gamma^{\sigma}\gamma^{\mu} + \cancel{\gamma^{\sigma}\gamma^{\rho}\gamma^{\nu}\gamma^{\mu} - \gamma^{\sigma}\gamma^{\rho}\gamma^{\nu}\gamma^{\mu}}\right)$$

But $2\gamma^{\nu}\gamma^{\rho} = \{\gamma^{\nu},\gamma^{\rho}\} + [\gamma^{\nu},\gamma^{\rho}]$ so

$$i\frac{i}{4}\Big(g^{\mu\sigma}\left(2g^{\nu\rho} + [\gamma^{\nu},\gamma^{\rho}]\right) - g^{\mu\rho}\left(2g^{\nu\sigma} + [\gamma^{\nu},\gamma^{\sigma}]\right) - g^{\nu\rho}\left(2g^{\mu\sigma} + [\gamma^{\sigma},\gamma^{\mu}]\right)$$
$$+ g^{\nu\sigma}\left(2g^{\mu\rho} + [\gamma^{\rho},\gamma^{\mu}]\right)\Big) =$$

$$= i\Big(\cancel{2g^{\mu\sigma}g^{\nu\rho}} + \underline{g^{\mu\sigma}S^{\nu\rho}} - \cancel{2g^{\mu\rho}g^{\nu\sigma}} - g^{\mu\rho}S^{\nu\sigma}$$
$$+ \cancel{2g^{\nu\sigma}g^{\mu\rho}} + g^{\nu\sigma}S^{\rho\mu} - \cancel{2g^{\nu\rho}g^{\mu\sigma}} + g^{\nu\rho}S^{\mu\sigma}\Big)$$

$$= i\left(g^{\mu\sigma}S^{\nu\rho} + g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho}\right) \quad \text{q.e.d.}$$

With CHIRAL REPRESENTATIONS FOR γ-matrices

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \text{which fulfills } \{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu} I_{4\times4}$$

i.e. $\gamma^{\mu} \equiv \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}$

$$S^{ij} = \frac{i}{4}[\gamma^i,\gamma^j] = \frac{i}{4}\left(\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\right) =$$

$$= \frac{i}{4}\left(\begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} - \begin{pmatrix} -\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i \end{pmatrix}\right) = \frac{i}{4}\begin{pmatrix} -(\sigma^i\sigma^j - \sigma^j\sigma^i) & 0 \\ 0 & -(\sigma^i\sigma^j - \sigma^j\sigma^i) \end{pmatrix} =$$

$$= -i\begin{pmatrix} i\varepsilon^{ijk}\frac{\sigma^k}{2} & 0 \\ 0 & i\varepsilon^{ijk}\frac{\sigma^k}{2} \end{pmatrix} = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\varepsilon^{ijk}\Sigma^k$$

$$S^{i0} = \frac{i}{4}[\gamma^i,\gamma^0] = \frac{i}{4}\left(\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} - \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\right) = \frac{i}{4}\left(\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} - \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\right)$$

$$= \frac{i}{4}\begin{pmatrix} 2\sigma^i & 0 \\ 0 & -2\sigma^i \end{pmatrix} = \frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

VARIATIONAL PRINCIPLE

Setting the problem: Consider a system whose states are specified by n field variables $\phi^\alpha(x)$ with $\alpha=\overline{1,n}$ and x are the spacetime coordinates

$x^\mu \to$ independent variables

$\phi^\alpha(x) \to$ state functions (real or complex; scalar, vector, spinor or tensor functions of spacetime coordinates)

Some definitions: Consider an infinitesimal transformation of independent variables

$$x'^\mu = x^\mu + \delta x^\mu(x)$$

Here $\delta(x^\mu) \equiv \lambda \xi^\mu(x)$ with $\lambda \to$ an infinitesimal parameter

$\xi^\mu(x) \to$ an arbitrary function of x

To a such infinitesimal variation of independent variables let us define

$\delta'\phi^\alpha(x) = \phi'^\alpha(x') - \phi^\alpha(x)$ → total variation of the field

$\delta\phi^\alpha(x) = \phi'^\alpha(x) - \phi^\alpha(x)$ → local field variation : change o a given field configuration at a fixed coordinate x

$\delta\phi^\alpha$ → compare fields variations at the same space-time point $\delta\phi^\alpha(x) = \lambda\eta^\alpha(x)$

Obs 1: $\underline{\delta'\phi^\alpha(x) = \delta\phi^\alpha(x) + \delta x^\mu \partial_\mu \phi^\alpha(x)}$ → Connection between total and local variations

Indeed $\delta'\phi^\alpha(x) = \phi'^\alpha(x') - \phi^\alpha(x) = \phi'^\alpha(x+\delta x) - \phi^\alpha(x) =$

$= \phi'^\alpha(x) + \underbrace{\delta x^\mu \partial_\mu \phi'(x)}_{\text{to first order}} - \phi^\alpha(x) =$

$= \delta\phi^\alpha(x) + \delta x^\mu \partial \phi(x)$

For partial derivative

$\delta'\partial_\mu\phi^\alpha \equiv \partial_\mu\phi'^\alpha(x') - \partial_\mu\phi^\alpha(x) = \partial_\mu(\phi^\alpha(x) + \delta\phi^\alpha(x) + \delta x^\nu\partial_\nu\phi^\alpha) - \partial_\mu\phi^\alpha(x) = \delta(\partial_\mu\phi^\alpha) + \delta x^\nu\partial_\mu\partial_\nu\phi^\alpha$

Obs 2: $\delta\phi(x') = \delta\phi(x+\delta x) \simeq \delta\phi(x) + \underset{\sim\lambda}{\delta x^\mu}\, \partial_\mu(\underbrace{\delta\phi(x)}_{\sim\lambda}) \simeq \delta\phi(x)$ to first order terms

Obs 3. $\underset{\substack{\text{local variation of}\\ \text{partial derivative}\\ \text{of the field}}}{\delta(\partial_\mu\phi^\alpha)} = \underbrace{\partial_\mu(\delta\phi^\alpha)}_{\substack{\text{partial derivative}\\ \text{of local field variation}}}$

$$\delta(\partial_\mu\phi^\alpha) = \partial_\mu\phi'^\alpha(x) - \partial_\mu\phi^\alpha(x) = \partial_\mu(\delta\phi^\alpha(x) + \phi^\alpha(x)) - \partial_\mu\phi^\alpha(x) = \partial_\mu(\delta\phi^\alpha(x))$$

• Suppose that the dynamics of the system is governed by the action (2N)

$$S = \int_R \mathcal{L}(x, \phi^\alpha, \partial_\mu \phi^\alpha)\, d^4x$$

where: R is a domain bounded by spacelike surfaces Σ_1 and Σ_2 which extends to infinity

$\mathcal{L}(x, \phi^\alpha, \partial_\mu \phi^\alpha) \to$ a scalar - Lagrangian density

It is assumed that the functional form of $\mathcal{L}$ is unchanged by the variations of the fields and independent variables x i.e.

$$\mathcal{L}'(x', \phi', \partial_\mu \phi') = \mathcal{L}(x', \phi', \partial_\mu \phi')$$

• The functional variation of the action is

$$\delta S \equiv \int_{R'} \mathcal{L}(x', \phi'^\alpha, \partial_\mu \phi'^\alpha)\, d^4x' - \int_R \mathcal{L}(x, \phi^\alpha, \partial_\mu \phi^\alpha)\, d^4x$$

and now we can reduce the integral over R' to an integral over the original region R. Since

$$d^4x' = \det\left(\frac{\partial x'}{\partial x}\right) d^4x \simeq \left(1 + \frac{\partial(\delta x^\mu)}{\partial x^\mu}\right) d^4x$$ to the first order in $\delta x^\mu(x)$

then the total variation is obtained by observing that

$$\mathcal{L}(x', \phi'^\alpha(x'), \partial_\mu \phi'^\alpha(x')) = \mathcal{L}(x^\mu + \delta x^\mu, \phi^\alpha(x) + \delta' \phi^\alpha(x), \partial_\mu \phi^\alpha(x) + \delta' \partial_\mu \phi^\alpha) =$$

$$= \mathcal{L}(x^\mu, \phi^\alpha(x), \partial_\mu \phi^\alpha(x)) + \frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi^\alpha}\delta' \phi^\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)}\delta'(\partial_\mu \phi^\alpha)$$

(i.e. now all quantities are expressed in terms of coordinates of the region R)

In conclusion

$$\delta S = \int_R \left(\mathcal{L}(x^\mu, \phi^\alpha(x), \partial_\mu \phi^\alpha(x)) + \frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi^\alpha}\delta' \phi^\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)}\delta'(\partial_\mu \phi^\alpha)\right)\left(1 + \frac{\partial(\delta x^\mu)}{\partial x^\mu}\right) d^4x -$$

$$- \int_R \mathcal{L}(x^\mu, \phi^\alpha(x), \partial_\mu \phi^\alpha(x))\, d^4x =$$

$$= \int_R \left(\mathcal{L}\frac{\partial(\delta x^\mu)}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi^\alpha}\delta' \phi^\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)}\delta'(\partial_\mu \phi^\alpha)\right) d^4x$$

Next step is to express all total variations in terms of local variations of the fields: (3N)

$$\delta S = \int_R \left(\mathcal{L} \frac{\partial(\delta x^\mu)}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi^\alpha}\left(\delta\phi^\alpha + \delta x^\mu \frac{\partial \phi^\alpha}{\partial x^\mu}\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)}\left(\delta(\partial_\mu \phi^\alpha) + \delta x^\nu \partial_\mu \partial_\nu \phi^\alpha\right)\right) d^4x =$$

$$= \int_R \left(\mathcal{L}\, \partial_\mu \delta x^\mu + \partial_\mu \mathcal{L}\, \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi^\alpha}\delta\phi^\alpha + \delta x^\mu \frac{\partial \mathcal{L}}{\partial \phi^\alpha}\partial_\mu \phi^\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^\alpha)}\partial_\mu \delta\phi^\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^\alpha)}\delta x^\nu \partial_\mu \partial_\nu \phi^\alpha \right) d^4x$$

Here we introduce the complete partial derivative: ie the partial derivative with respect to the independent variables when also state variables ϕ^α are seen as functions of independent variables.

$$\mathcal{D}_\mu = \frac{\partial}{\partial x^\mu} + \partial_\mu \phi^\alpha \frac{\partial}{\partial \phi^\alpha} + \partial_\mu \partial_\nu \phi^\alpha \frac{\partial}{\partial(\partial_\nu \phi^\alpha)}$$

Also observe that

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)}\delta(\partial_\mu \phi^\alpha) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^\alpha)}\partial_\mu(\delta\phi^\alpha) = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^\alpha)}\delta\phi^\alpha\right) - \delta\phi^\alpha \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)}\right) =$$

Therefore

$$= \mathcal{D}_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^\alpha)}\delta\phi^\alpha\right) - \delta\phi^\alpha \mathcal{D}_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\alpha)}\right)$$

$$\delta S = \int_R \Big(\mathcal{L}\,\partial_\mu \delta x^\mu + \delta x^\mu \underline{\frac{\partial \mathcal{L}}{\partial x^\mu}} + \underline{\frac{\partial \mathcal{L}}{\partial \phi^\alpha}}\delta\phi^\alpha + \underline{\delta x^\mu \frac{\partial \mathcal{L}}{\partial \phi^\alpha}\partial_\mu \phi^\alpha} - \underline{\delta\phi^\alpha \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^\alpha)}\right)} + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^\alpha)}\delta\phi^\alpha\right) +$$

$$+ \underline{\delta x^\mu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^\alpha)}\partial_\nu \partial_\mu \phi^\alpha} \Big) d^4x$$

$$= \int_R \left[\left(\frac{\partial \mathcal{L}}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial \phi^\alpha}\partial_\mu \phi^\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\nu\phi^\alpha)}\partial_\nu \partial_\mu \phi^\alpha\right)\delta x^\mu + \mathcal{L}\frac{\partial \delta x^\mu}{\partial x^\mu} + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^\alpha)}\delta\phi^\alpha\right) + \right.$$

$$\left. + \left(\frac{\partial \mathcal{L}}{\partial \phi^\alpha} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^\alpha)}\right)\delta\phi^\alpha \right] d^4x$$

$$= \int_R \left[\mathcal{D}_\mu\left(\underline{\mathcal{L}\,\delta x^\mu}\right) + \mathcal{D}_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)}\delta\phi^\alpha\right) + \left(\frac{\partial \mathcal{L}}{\partial \phi^\alpha} - \mathcal{D}_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^\alpha)}\right)\right)\delta\phi^\alpha\right] d^4x$$

$$= \int_R \left[\mathcal{D}_\mu\left(\mathcal{L}\,\delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^\alpha)}\delta\phi^\alpha\right) + \mathcal{E}^\alpha \delta\phi^\alpha\right] d^4x$$

CLASSICAL WEYL FIELDS

Let us introduce the Pauli matrices $\vec{\sigma}$ as well as

$$\sigma = (I, \vec{\sigma}) \equiv (\sigma^\mu)$$
$$\bar{\sigma} = (I, -\vec{\sigma}) \equiv (\bar{\sigma}^\mu)$$

Comment $\sigma^\mu = (\sigma^\mu)_{\alpha\dot{\alpha}}$

$\bar{\sigma}^\mu = (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}$

$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$

and note the properties: $\vec{\sigma}^* = -\sigma_2 \vec{\sigma} \sigma_2$

$$(\bar{\sigma}^\mu)^* = (I, -\vec{\sigma}^*) = (\sigma_2\sigma_2, \sigma_2\vec{\sigma}\sigma_2) = \sigma_2(\sigma^\mu)\sigma_2$$
$$(\sigma^\mu)^* = (I, \vec{\sigma}^*) = (\sigma_2\sigma_2, -\sigma_2\vec{\sigma}\sigma_2) = \sigma_2(\bar{\sigma}^\mu)\sigma_2$$

Let be $\psi_L(x)$ – the left-handed Weyl field

At a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ $\underline{\psi_L^+ \bar{\sigma}^\mu \psi_L}$ transform as a four-vector.

Proof. Since $U_D^{-1}\gamma^\mu U_D = \Lambda^\mu{}_\nu \gamma^\nu$; Here $U_D = \begin{pmatrix} U_L & 0 \\ 0 & U_R \end{pmatrix}$

$$\begin{pmatrix} U_L^{-1} & 0 \\ 0 & U_R^{-1} \end{pmatrix}\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}\begin{pmatrix} U_L & 0 \\ 0 & U_R \end{pmatrix} = \Lambda^i{}_\nu \gamma^\nu$$
$$\begin{pmatrix} U_L^{-1} & 0 \\ 0 & U_R^{-1} \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} U_L & 0 \\ 0 & U_R \end{pmatrix} = \Lambda^0{}_\nu \gamma^\nu$$
$$\Rightarrow \begin{pmatrix} U_L^{-1} & 0 \\ 0 & U_R^{-1} \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} U_L & 0 \\ 0 & U_R \end{pmatrix} = \Lambda^\mu{}_\nu \gamma^\nu$$

or

$$\begin{pmatrix} 0 & U_L^{-1}\sigma^\mu \\ U_R^{-1}\bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} U_L & 0 \\ 0 & U_R \end{pmatrix} = \begin{pmatrix} 0 & \Lambda^\mu{}_\nu\sigma^\nu \\ \Lambda^\mu{}_\nu\bar{\sigma}^\nu & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & \underline{U_L^{-1}\sigma^\mu U_R} \\ U_R^{-1}\bar{\sigma}^\mu U_L & 0 \end{pmatrix} = \begin{pmatrix} 0 & \underline{\Lambda^\mu{}_\nu\sigma^\nu} \\ \Lambda^\mu{}_\nu\bar{\sigma}^\nu & 0 \end{pmatrix}$$

and since $U_L^{-1} = U_R^+$; $U_R^{-1} = U_L^+$ we conclude that

$$\begin{cases} U_R^+ \sigma^\mu U_R = \Lambda^\mu{}_\nu \sigma^\nu \\ U_L^+ \bar{\sigma}^\mu U_L = \Lambda^\mu{}_\nu \bar{\sigma}^\nu \end{cases}$$

Then $\overline{(\psi_L^+ \bar{\sigma}^\mu \psi_L)'} = \underbrace{\underline{\psi_L^+ U_L^+} \bar{\sigma}^\mu \underline{U_L \psi_L}} = \Lambda^\mu{}_\nu \overline{(\psi_L^+ \bar{\sigma}^\mu \psi_L)}$ q.e.d.

Analogously

$$(\psi_R^+ \sigma^\mu \psi_R)' = \psi_R^+ U_R^+ \sigma^\mu U_R \psi_R = \Lambda^\mu{}_\nu (\psi_R^+ \sigma^\mu \psi_R)$$

Other Lorentz invariants constructions

① $\Psi_L^+ \Psi_R$ is an invariant at Lorentz transformation

Proof $(\Psi_L^+\Psi_R)' = \Psi_L'^+ \Psi_R' = (U_L \Psi_L)^+ U_R \Psi_R = \Psi_L^+ U_L^+ U_R \Psi_R = \Psi_L^+ U_L^+ (U_L^+)^{-1} \Psi_R = \Psi_L^+ \Psi_R$

② $\Psi_R^+ \Psi_L$ → is Lorentz invariant

Proof $(\Psi_R^+ \Psi_L)' = \Psi_R'^+ \Psi_L' = (U_R \Psi_R)^+ U_L \Psi_L = \Psi_R^+ U_R^+ U_L \Psi_L = \Psi_R^+ U_R^+ (U_R^+)^{-1} \Psi_L = \Psi_R^+ \Psi_L$

③ $(-i\sigma_2 \Psi_R)^T \Psi_R$ → is Lorentz invariant

Proof: $\left((-i\sigma_2 \Psi_R)^T \Psi_R\right)' = -(i\sigma_2 \Psi_R')^T \Psi_R' = -(i\sigma_2 U_R \Psi_R)^T U_R \Psi_R = -i \Psi_R^T U_R^T \sigma_2^T U_R \Psi_R =$

$$= \underbrace{-i\Psi_R^T \sigma_2^T}\, \underbrace{\sigma_2^T U_R^T \sigma_2^T}_{(U_R^T)^{-1}} U_R \Psi_R = (-i\sigma_2 \Psi_R)^T \underbrace{(U_R^T)^{-1} U_R}_{I} \Psi_R =$$

$$= (-i\sigma_2 \Psi_R)^T \Psi_R \quad \text{q.e.d}$$

Equivalent writing: $\xi_{\dot\alpha} \eta^{\dot\alpha} = (\xi_{\dot 1}\ \xi_{\dot 2}) \begin{pmatrix} \eta^{\dot 1} \\ \eta^{\dot 2} \end{pmatrix} = \xi_{\dot 1} \eta^{\dot 1} + \xi_{\dot 2} \eta^{\dot 2}$ is a Lorentz invariant

④ $\left((+i\sigma_2 \Psi_L)^T \Psi_L\right)' = (i\sigma_2 \Psi_L)^T \Psi_L$ is a Lorentz invariant

Equivalent $\chi^\alpha \eta_\alpha = (\chi^1\ \chi^2) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \chi^1 \eta_1 + \chi^2 \eta_2 = \epsilon^{\alpha\beta} \chi_\alpha \eta_\beta$

For DIRAC SPINORS

$\Psi_D = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$; since $\bar\Psi_D \Psi_D = (\Psi_L^+\ \Psi_R^+) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = (\Psi_L^+\ \Psi_R^+) \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix} = \Psi_R^+ \Psi_L + \Psi_L^+ \Psi_R$

we may conclude that (see above) this product is Lorentz inv.

Equivalently: $\Psi_D = \begin{pmatrix} \chi_\alpha \\ \xi^{\dot\beta} \end{pmatrix} \Rightarrow \bar\Psi_D \Psi_D = (\chi_\alpha^+\ \xi^{\dot\beta +}) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \chi_\alpha \\ \xi^{\dot\beta} \end{pmatrix} = (\xi^{\dot\beta +}\ \chi_\alpha^+) \begin{pmatrix} \chi_\alpha \\ \xi^{\dot\beta} \end{pmatrix}$

$$= \xi^\alpha \chi_\alpha + \chi_{\dot\alpha} \xi^{\dot\alpha} \qquad \xi^{\dot\beta +} \to \xi^\alpha, \quad \chi_\alpha^+ \to \chi_{\dot\alpha}$$

(here ξ^α is obtained from $\xi^{\dot\alpha}$ by complex conjugation) since $(\xi^{\dot\alpha})^*$ transform with U_R^* i.e. is ξ^α-type

① The Lagrangian density for Left-handed WEYL field (3v)

With ψ_L and ψ_L^* independent fields we construct the kinetic term:

$$\mathcal{L}_L = i\,\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L$$

By construction this quantity is Lorentz invariant:

Proof

$$\mathcal{L}_L' = i\,\psi_L'^\dagger \bar{\sigma}^\mu \partial_\mu' \psi_L' = i\,\psi_L^\dagger U_L^\dagger \bar{\sigma}^\mu \Lambda_\mu{}^\nu \partial_\nu U_L \psi_L = i\,\psi_L^\dagger U_L^\dagger \bar{\sigma}^\mu U_L \Lambda_\mu{}^\nu \partial_\nu \psi_L =$$

$$= i\,\psi_L^\dagger \Lambda^\mu{}_\rho \bar{\sigma}^\rho \Lambda_\mu{}^\nu \partial_\nu \psi_L = i\,\psi_L^\dagger \delta_\rho^\nu \bar{\sigma}^\rho \partial_\nu \psi_L = i\,\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L \quad \text{q.e.d.}$$

By construction $\mathcal{L}_L$ contains a factor "i" in front, fixed by the condition that the action $\int d^4x\, \mathcal{L}_L$ is real

② Euler-Lagrange equations

Since $\mathcal{L}_L = i\,\psi_{L\alpha}^* \bar{\sigma}^\mu_{\alpha\beta} \partial_\mu \psi_{L\beta}$ we can choose $\psi_{L\alpha}$ and $\psi_{L\beta}^*$ independent fields

Then

$$\frac{\partial \mathcal{L}}{\partial \psi_{L\gamma}^*} = i\,\delta_{\alpha\gamma} \bar{\sigma}^\mu_{\alpha\beta} \partial_\mu (\psi_{L\beta}) = i\,\bar{\sigma}^\mu \partial_\mu \psi_L$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_{L\gamma}^*)} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \psi_L^*} = 0 \quad \text{and so}$$

$$\boxed{i\,\bar{\sigma}^\mu \partial_\mu \psi_L = 0} \rightarrow \text{First-order differential equation}$$

Equivalently: $i(\partial_0 \psi_L - \vec{\sigma}\cdot\vec{\nabla}\psi_L) = 0 \Rightarrow \boxed{(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = 0}$ WEYL EQUATION

Observation: Each component of ψ_L satisfy Klein-Gordon equation for mass zero. Indeed, if we apply $\sigma^\nu\partial_\nu$ results

$$\sigma^\nu \partial_\nu \bar{\sigma}^\mu \partial_\mu \psi_L = 0 \Rightarrow \sigma^\nu \bar{\sigma}^\mu \partial_\nu \partial_\mu \psi_L = 0 \Rightarrow \partial_0^2 \psi_L - \sigma^i \sigma^j \partial_i \partial_j \psi_L = 0$$

$$\Rightarrow \partial_0^2 \psi_L - (\delta^{ij} + i\varepsilon^{ijk}\sigma^k)\partial_i \partial_j \psi_L = 0 \Rightarrow$$

$$\Rightarrow \partial_0^2 \psi_L - \partial_i^2 \psi_L - \underbrace{i\varepsilon^{ijk}\sigma^k \partial_i \partial_j \psi_L}_{0} = 0 \Rightarrow (\partial_0 \partial^0 + \partial_i \partial^i)\psi_L = 0 \Rightarrow$$

$$\partial_\nu \partial^\nu \psi = 0 \quad \text{or} \quad \boxed{\Box \psi = 0} \qquad \text{with } \Box = \partial_0^2 - (\text{grad})^2 = \partial_0^2 - \vec{\nabla}^2$$

Analogously, observe that

(LW)

$$\left.\begin{array}{l}\dfrac{\partial \mathcal{L}}{\partial \psi_{L\gamma}} = 0 \\ \\ \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_{L\gamma})} = i\,\psi^*_{L\alpha}\,\bar{\sigma}^\mu_{\alpha\beta}\,\delta_{\gamma\beta} \longrightarrow i\psi^+_L\bar{\sigma}^\mu\end{array}\right\} \overset{E-L}{\Longrightarrow} i\,\partial_\mu \psi^+_L\,\bar{\sigma}^\mu = 0$$

and the momentum canonically conjugated quantities are

to ψ_L $\quad \dfrac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi_L}{\partial t}\right)} = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \psi_L)} = i\,\psi^+_L\bar{\sigma}^0 = i\,\psi^+_L$

to ψ^*_L $\quad \dfrac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi^*_{L\gamma}}{\partial t}\right)} = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \psi^*_{L\gamma})} = 0$!!!

The Hamiltonian density:

$$\mathcal{H}(x) = \Pi_{\psi_L}\frac{\partial \psi_L}{\partial t} - \mathcal{L} = i\,\psi^+_L\bar{\sigma}^0\frac{\partial \psi_L}{\partial t} - i\,\psi^+_L\bar{\sigma}^\mu\partial_\mu\psi_L =$$

$$= \cancel{i\,\psi^+_L\frac{\partial \psi_L}{\partial t}} - \cancel{i\,\psi^+_L\frac{\partial \psi_L}{\partial t}} + i\,\psi^+_L(\vec{\sigma})^i\partial_i\psi_L =$$

$$= i\,\psi^+_L\vec{\sigma}\cdot\vec{\nabla}\psi_L \underset{\substack{\uparrow \\ \text{from eq of} \\ \text{motion}}}{=}$$

and $\quad H = \int d^3x\; i\,\psi^+_L\,\vec{\sigma}\cdot\vec{\nabla}\,\psi_L$

4. Dynamical invariants for left-handed Weyl fields

5W

For the invariance to space-time translations: the Noether current is the stress-energy tensor (observe that for ψ_L which verify the eq E-L $\bar{\sigma}^\mu \partial_\mu \psi_L = 0 \Rightarrow \mathcal{L} = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L = 0$

$$\Theta^{\mu\nu}(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_L)} \partial^\nu \psi_L - g^{\mu\nu}\mathcal{L} = i\psi_L^\dagger \bar{\sigma}^\mu \partial^\nu \psi_L$$

ψ_L^*, ψ_L

For the invariance to Lorentz transformations:

$$\mathcal{M}^{\mu(\rho\eta)} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_{L\alpha})} \left(I^{(\rho\eta)}\right)^\alpha{}_\beta \psi_\beta + \underset{\Rightarrow}{0} + \Theta^{\mu\eta} x^\rho - \Theta^{\mu\rho} x^\eta$$

Now $\left(I^{(\rho\eta)}\right)^\alpha{}_\beta = -i\left(J_L^{\rho\eta}\right)^\alpha{}_\beta$ where $J_L^{\rho\eta} = \frac{i}{4}\left(\sigma^\rho \bar{\sigma}^\eta - \sigma^\eta \bar{\sigma}^\rho\right)$

Then

$$\mathcal{M}^{\mu(\rho\eta)} = i\psi_L^\dagger \bar{\sigma}^\mu (-i)\frac{i}{4}\left(\sigma^\rho\bar{\sigma}^\eta - \sigma^\eta\bar{\sigma}^\rho\right)\psi_L + i\psi_L^\dagger \bar{\sigma}^\mu\left(x^\rho\partial^\eta - x^\eta\partial^\rho\right)\psi_L$$

$$= \frac{i}{4}\psi_L^\dagger \bar{\sigma}^\mu\left(\sigma^\rho\bar{\sigma}^\eta - \sigma^\eta\bar{\sigma}^\rho\right)\psi_L + i\psi_L^\dagger\bar{\sigma}^\mu\left(x^\rho\partial^\eta - x^\eta\partial^\rho\right)\psi_L$$

In conclusion

$$\mathcal{M}^{\mu(\rho\eta)}(x) = i\psi_L^\dagger\bar{\sigma}^\mu\left(x^\rho\partial^\eta - x^\eta\partial^\rho + \frac{1}{4}\left(\sigma^\rho\bar{\sigma}^\eta - \sigma^\eta\bar{\sigma}^\rho\right)\right)\psi_L$$

Now again

$$M^{\rho\eta} = \int d^3x\, \mathcal{M}^{0\rho\eta}$$

$$\vec{J} = \frac{\vec{e}_i}{2}\varepsilon^{ijk}M^{jk} = \frac{\vec{e}_i}{2}\varepsilon^{ijk}\int d^3x\, i\psi_L^\dagger \overset{\overset{I}{\uparrow}}{\bar{\sigma}^0}\left(x^j\partial^k - x^k\partial^j + \frac{1}{4}\left(\sigma^j\bar{\sigma}^k - \sigma^k\bar{\sigma}^j\right)\right)\psi_L$$

$$= \vec{e}_i \frac{i}{2}2\int d^3x\,\psi_L^\dagger\left(\vec{r}\times(-\vec{\nabla})\right)^i\psi_L + \frac{\vec{e}_i}{2}i\int\varepsilon^{ijk}\psi_L^\dagger\frac{2(-)\sigma^j\sigma^k}{4}\psi\, d^3x$$

$$= \vec{e}_i\int d^3x\,\psi_L^\dagger\left(\vec{r}\times\frac{1}{i}\vec{\nabla}\right)^i\psi_L + \frac{\vec{e}_i}{2}\int d^3x\,\psi_L^\dagger\, i(-)\varepsilon^{ijk}\frac{1}{2}i\varepsilon^{jkl}\sigma^l\psi_L$$

Then

$$\vec{L} = \int d^3x\,\psi_L^\dagger\left(\vec{r}\times\frac{1}{i}\vec{\nabla}\right)\psi_L$$

$$\vec{S} = \frac{1}{2}\int d^3x\,\psi_L^\dagger\,\vec{\sigma}\,\psi_L$$

The invariance under a global $U(1)$ internal ~~symmetry~~ transformation (6W)

$$\psi_L \to e^{i\theta}\psi_L$$

leads to the Noether current

$$j^\mu = \psi_L^\dagger \bar{\sigma}^\mu \psi_L$$

WEYL FIELDS: POSITIVE AND NEGATIVE FREQUENCY COMPONENTS (7W)

$$\psi_L(x) = \frac{1}{(2\pi)^2}\int d^4k\, e^{ik_\mu x^\mu}\tilde{\psi}_L(k) \xrightarrow[\text{case}]{\text{see scalar field}} \frac{1}{(2\pi)^2}\sqrt{2\pi}\int d^4k\, e^{ikx}\delta(k^2)\psi_L(k) =$$

$$= \frac{1}{(2\pi)^{3/2}}\int d^4k\, e^{ikx}\delta(k^2)\left[\theta(k_0)+\theta(-k_0)\right]\psi_L(k) =$$

$$= \frac{1}{(2\pi)^{3/2}}\int d^4k\, e^{ik_\mu x^\mu}\delta(k_0^2-\vec{k}^2)\,\theta(k_0)\,\psi_L(k) + \frac{1}{(2\pi)^{3/2}}\int d^4k\, e^{ik_\mu x^\mu}\delta(k_0^2-\vec{k}^2)\,\theta(-k_0)\psi_L(k)$$

$$= \psi_L^+(x) + \psi_L^-(x)$$

Now

$$\psi_L^+(x) = \frac{1}{(2\pi)^{3/2}}\int dk_0 d^3k\, e^{ik_\mu x^\mu}\delta(k_0^2-\vec{k}^2)\theta(k_0)\psi_L(k) = \frac{1}{(2\pi)^{3/2}}\int dk_0 d^3k\, e^{ik_\mu x^\mu}\theta(k_0)\frac{\left(\delta(k_0-\sqrt{\vec{k}^2})+\delta(k_0+\sqrt{\vec{k}^2})\right)}{2\sqrt{\vec{k}^2}}\psi_L(k)$$

$$\omega_p = \sqrt{\vec{k}^2} \qquad = \frac{1}{(2\pi)^{3/2}}\int d^3k\, e^{i(\omega_p x^0 - \vec{k}\cdot\vec{x})}\frac{\psi_L(k)}{2\omega_p} = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{\sqrt{2\omega_p}}e^{i(\omega_p x^0 - \vec{k}\cdot\vec{x})}\psi_L^+(\vec{k})$$

with $\psi_L^+(\vec{k}) = \dfrac{\psi(k)}{\sqrt{2\omega_p}}$

Analogously $\psi_L^-(x) = \frac{1}{(2\pi)^{3/2}}\int dk_0 d^3k\, e^{ik_\mu x^\mu}\theta(-k_0)\frac{\delta(k_0-\sqrt{\vec{k}^2})+\delta(k_0+\sqrt{\vec{k}^2})}{2\sqrt{\vec{k}^2}}\psi_L(k)$ $\quad \xrightarrow[\vec{k}\to-\vec{k}]{k_0\to -k_0}$

$$\psi_L^-(x) = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{\sqrt{2\omega_p}}e^{-i(\omega_p x^0-\vec{k}\cdot\vec{x})}\psi_L^-(\vec{k}) \qquad \text{with } \psi_L^-(\vec{k}) = \frac{\psi_L(-k)}{\sqrt{2\omega_p}}$$

Now from Euler-Lagrange equation $i\bar{\sigma}^\mu\partial_\mu\psi_L(x)=0$ is deduced $(\partial_0 - \sigma^i\partial_i)\psi_L = 0$

$$\frac{1}{(2\pi)^2}\int d^4k\, i\bar{\sigma}^\mu ik_\mu e^{ik_\mu x^\mu}\sqrt{2\pi}\,\delta(k^2)\psi_L(k) = 0 \Rightarrow \bar{\sigma}^\mu k_\mu\psi_L(k) = 0 \Rightarrow (Ik^0 + \vec{\sigma}\cdot\vec{k})\psi_L(k) = 0$$

$$\Rightarrow \boxed{\frac{\vec{\sigma}\cdot\vec{k}}{k^0}\psi_L(k) = -\psi_L(k)}$$

$$\frac{\vec{\sigma}\cdot\vec{k}}{\omega_p}\psi_L(k) = -\psi_L(k) \iff \frac{\vec{p}\cdot\vec{J}}{\omega_p}\psi_L = -\frac{1}{2}\psi_L(\vec{k}) \rightarrow$$ a left-handed massless Weyl spinor has helicity $h = -\frac{1}{2}$

$$\frac{\vec{\sigma}\cdot\vec{k}}{\omega_p}\psi_L(k) = -\psi_L(k) \Rightarrow \frac{\vec{\sigma}\cdot\vec{k}}{\omega_p}\psi_L^+(\vec{k}) = -\psi_L^+(\vec{k}) \Rightarrow v_L^\nu(\vec{k}) \Rightarrow$$

$\Downarrow$ $\qquad$ ν associated to helicity $-\frac{1}{2}$ $\Rightarrow \psi_L^+(\vec{k}) = v_L^\nu(\vec{k})\, b_\nu^+(\vec{k})$

$$\frac{\vec{\sigma}(-\vec{k})}{-\omega_p}\psi_L(-k) = -\psi_L(-k) \Rightarrow \frac{\vec{\sigma}\cdot\vec{k}}{\omega_p}\psi_L^-(\vec{k}) = -\psi_L^-(\vec{k}) \Rightarrow u_L^\nu(\vec{k})$$

$$\psi_L^-(\vec{k}) = u_L^\nu(\vec{k})\, a^-(\vec{k})$$

REAL SCALAR FIELD $\phi(x)$

$\phi'(x') = \phi(x)$ when $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ $\quad g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

1. Construction of the action

$$\partial_\mu \phi \equiv \frac{\partial \phi}{\partial x^\mu} \qquad \partial_\mu \phi\, \partial^\mu \phi \to \text{a kinetic term Lorentz invariant}$$

$$S = \int d^4x \, \frac{1}{2}\left(\partial^\mu \phi\, \partial_\mu \phi - m^2 \phi^2\right) \Rightarrow \mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

2. Euler - Lagrange equations $\to$ Klein - Gordon equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

But:

$$\frac{\partial \mathcal{L}}{\partial \phi} = - m^2 \phi$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = \frac{1}{2}\partial_\mu \frac{\partial}{\partial(\partial_\mu \phi)}\left(g^{\eta\nu} \partial_\eta \phi\, \partial_\nu \phi\right) = \frac{1}{2}\partial_\mu \left(g^{\eta\nu}\delta_{\eta\mu}\partial_\nu \phi + g^{\eta\nu}\partial_\eta \phi\, \delta_{\nu\mu}\right)$$
$$= \partial_\mu \left(\partial^\mu \phi\right)$$

$$\Rightarrow \quad \left(\partial_\mu \partial^\mu + m^2\right)\phi(x) = 0 \qquad \to \text{Klein - Gordon equation}$$

Equiv. form with $\Box = -\partial_\nu \partial^\nu = \nabla^2 - \frac{\partial^2}{\partial t^2}$ $\qquad (\Box - m^2)\phi = 0$

3. Momentum conjugate and the Hamiltonian

$$\Pi_\phi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi = \partial_0 \phi = \dot{\phi}$$

Legendre transformation $\Rightarrow$

$$\mathcal{H} = \Pi_\phi\, \partial_0 \phi - \mathcal{L} = \Pi_\phi^2 - \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\mathrm{grad}\,\phi)^2 + \frac{1}{2}m^2\phi^2 =$$

$$= \underbrace{\frac{1}{2}\Pi_\phi^2}_{\text{energy cost of moving in time}} + \underbrace{\frac{1}{2}(\nabla \phi)^2}_{\text{of shearing in space}} + \underbrace{\frac{1}{2}m^2\phi^2}_{\text{energy cost of having the field around at all}}$$

Obs $\mathcal{H}$ is positive definite and this justify the signs choice in $\mathcal{L}$

4. Canonical energy-momentum stress tensor

$$\Theta^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^{\mu}{}_{\nu} \mathcal{L}$$

or in our case, with $\Theta^{\mu\nu} = g^{\nu\eta} \Theta^{\mu}{}_{\eta}$

$$\Rightarrow \quad \Theta^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}$$

$$\Theta^{00} = \partial^0 \phi \partial^0 \phi - \mathcal{L} = \frac{1}{2} (\partial^0 \phi)^2 + \frac{1}{2} (\text{grad}\, \phi)^2 + \frac{1}{2} m^2 \phi^2 = \mathcal{H} \Rightarrow$$

The associated conserved „charge":

$$H = \int d^3x\, \Theta^{00} \quad \rightarrow \text{ total energy of the field}$$

$$\Theta^{0i} = \partial^0 \phi \partial^i \phi = \Pi \partial^i \phi = -\Pi \partial_i \phi$$

The conserved „charge":

$$P^i = \int d^3x\, \Theta^{0i} = -\int \Pi \partial_i \phi \, d^3x \quad \rightarrow \text{ total linear momentum carried by the field.}$$

5. Angular momentum

Consider a Lorentz transformation: $\delta x^\mu = \omega^\mu{}_\nu x^\nu$ (infinitesimal $\omega^\mu{}_\nu \ll 1$)

$$\delta x^\mu = \omega^\mu{}_\nu x^\nu = \sum_{\rho<\sigma} \omega^{\rho\sigma} \left(\delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho \right)$$

$(\rho\sigma) \longleftrightarrow \kappa \qquad \omega^{\rho\sigma} \leftrightarrow \epsilon^\kappa$ (see the discussion at Noether theorem)

$$\Gamma^\mu{}_{(\rho\sigma)} = \frac{\partial(\delta x^\mu)}{\partial \omega^{\rho\sigma}} = \delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho \Rightarrow \Gamma^{\mu(\rho\sigma)} = g^{\mu\rho} x^\sigma - g^{\mu\sigma} x^\rho$$

$$\phi'(x') = \phi(x) \Rightarrow \Delta\phi = 0 \Rightarrow G^{(\rho\sigma)} = \frac{\partial(\Delta\phi)}{\partial \omega_{\rho\sigma}} = 0 .$$

Then the conserved Noether current is

$$\mathcal{M}^{\mu(\rho\sigma)}(x) = \Pi^\mu G^{(\rho\sigma)}(x,\phi) - \Theta^\mu{}_\nu \Gamma^{\nu(\rho\sigma)} = 0 - \Theta^\mu{}_\nu \left(g^{\nu\rho} x^\sigma - g^{\nu\sigma} x^\rho \right) \text{ or}$$

$$\mathcal{M}^{\mu(\rho\sigma)} = x^\rho \Theta^{\mu\sigma} - x^\sigma \Theta^{\mu\rho}$$

The conserved charge associated the spatial rotation is orbital angular momentum carried by the field (no spin)

$\mu=0$
$\rho\sigma = ij$

$$M^{ij} = \int d^3x \left(x^i \Theta^{0j} - x^j \Theta^{0i} \right) = \int d^3x\, \Pi \left(x^i \partial^j - x^j \partial^i \right) \phi$$

6. Momentum representation. Positive and negative frequency decomposition

Step 1. Fourier decomposition:

$$\phi(x) = \frac{1}{(2\pi)^2}\int d^4k\, e^{ik_\mu x^\mu}\tilde{\phi}(k) \quad \text{with} \quad k_\mu = (k_0, k_i) \rightarrow \text{four-momentum}$$

$k^2 = k_\mu k^\mu = g_{\mu\nu}k^\mu k^\nu = k^{0\,2} - \vec{k}^2$

$d^4k = dk_0\, d^3\vec{k}$

Obs: If $\phi(x)$ is real $\phi(x) = \phi^*(x)$ then $\frac{1}{(2\pi)^2}\int d^4k\,(\tilde{\phi}^*(k) - \tilde{\phi}(-k))e^{ik_\mu x^\mu} = 0$
so we have a condition on fourier components of the field $\tilde{\phi}^*(k) = \tilde{\phi}(-k)$

Since $\partial_\mu\partial^\mu\phi = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\phi = g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\phi = \frac{1}{(2\pi)^2}\int d^4k\, g^{\mu\nu}(ik_\mu ik_\nu)e^{ik_\sigma x^\sigma}\tilde{\phi}(k)$

$$= -\frac{1}{(2\pi)^2}\int d^4k\, k_\mu k^\mu e^{ik_\sigma x^\sigma}\tilde{\phi}(k) = -\frac{1}{(2\pi)^2}\int d^4k\, k^2 e^{ik_\mu x^\mu}\tilde{\phi}(k)$$

KG eq $\Rightarrow$ $\partial_\mu\partial^\mu\phi + m^2\phi = 0 \Rightarrow -\frac{1}{(2\pi)^2}\int d^4k\,(k^2 - m^2)e^{ik_\mu x^\mu}\tilde{\phi}(k) = 0$

and therefore $\tilde{\phi}(k)$ should verify:

$$(k^2 - m^2)\tilde{\phi}(k) = 0 \Rightarrow \tilde{\phi}(k) = \sqrt{2\pi}\,\delta(k^2 - m^2)\phi(k)$$

$\tilde{\phi}(k)$ is nonzero provided $k^2 = m^2$ or $(k^0)^2 - (\vec{k})^2 = m^2 \Rightarrow \boxed{k^{0\,2} = m^2 + \vec{k}^2}$

So

$$\phi(x) = \frac{1}{(2\pi)^{3/2}}\int d^4k\,\delta(k^2 - m^2)e^{ikx}\phi(k)$$

Step 2. The decomposition of the field in a positive frequency part and a negative frequency part:

- introduce $\theta(k_0) = \begin{cases} 1 & \text{if } k_0 \geq 0 \\ 0 & \text{if } k_0 < 0 \end{cases}$

$$\phi(x) = \frac{1}{(2\pi)^{3/2}}\int d^4k\, e^{ikx}\delta(k^2 - m^2)\left[\theta(k_0) + \theta(-k_0)\right]\phi(k) =$$

$$= \frac{1}{(2\pi)^{3/2}}\int dk_0\, d^3\vec{k}\, e^{ikx}\delta(k^2 - m^2)\theta(k_0)\phi(k) + \frac{1}{(2\pi)^{3/2}}\int dk_0\, d^3\vec{k}\, e^{ikx}\theta(-k_0)\phi(k) =$$

$$= \frac{1}{(2\pi)^{3/2}}\int dk_0\, d^3\vec{k}\, e^{ikx}\delta(k^2 - m^2)\theta(k_0)\phi(k) + \frac{1}{(2\pi)^{3/2}}\int dk_0\, d^3\vec{k}\, e^{-ikx}\theta(k_0)\phi(-k)$$

$$\equiv \phi^+(x) + \phi^-(x)$$

$\phi^+(x) \rightarrow$ positive frequency part of field $\phi(x)$

$\phi^-(x) \rightarrow$ negative frequency part of field $\phi(x)$

Step 3. Reduction to the integral over three-momentum $\vec{k}$

A δ-function property $\delta(xy) = \frac{1}{|x|}\delta(y) + \frac{1}{|y|}\delta(x)$

Then $\delta(k^2 - m^2) = \delta(k_0^2 - \vec{k}^2 - m^2) = \delta((k_0 - \sqrt{\vec{k}^2+m^2})(k_0 + \sqrt{\vec{k}^2+m^2})) =$

$$= \frac{1}{2\sqrt{\vec{k}^2+m^2}}\left(\delta\left(k_0 - \sqrt{\vec{k}^2+m^2}\right) + \delta\left(k_0 + \sqrt{\vec{k}^2+m^2}\right)\right)$$

Therefore:

$$\phi^+(x) = \frac{1}{(2\pi)^{3/2}}\int dk_0 d^3\vec{k}\, e^{ikx}\,\delta(k_0^2 - (\vec{k}^2 + m^2))\,\theta(k_0)\phi(k) =$$

$$= \frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}\, dk_0\, e^{ik_\mu x^\mu} \frac{1}{2\sqrt{\vec{k}^2+m^2}}\left(\delta(k_0 - \sqrt{\vec{k}^2+m^2}) + \delta(k_0 + \sqrt{\vec{k}^2+m^2})\right)\theta(k_0)\phi(k) =$$

$$= \frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}\, e^{i(\sqrt{\vec{k}^2+m^2}\,x^0 - \vec{k}\cdot\vec{x})} \frac{1}{2\sqrt{\vec{k}^2+m^2}}\,\phi\left(\sqrt{\vec{k}^2+m^2}, \vec{k}\right) =$$

With $\omega_p = +\sqrt{\vec{k}^2+m^2}$

$$= \frac{1}{(2\pi)^{3/2}}\int \frac{d^3\vec{k}}{\sqrt{2\omega_p}}\, e^{i(\omega_p x^0 - \vec{k}\cdot\vec{x})}\,\frac{\phi(\omega_p, \vec{k})}{\sqrt{2\omega_p}} = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3\vec{k}}{\sqrt{2\omega_p}}\, e^{i(\omega_p x^0 - \vec{k}\cdot\vec{x})}\,\phi^+(\vec{k})$$

where $\phi^+(\vec{k}) \equiv \frac{\phi(\omega_p, \vec{k})}{\sqrt{2\omega_p}}$ and ω_p defined above.

Conclusion:

$$\phi^+(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3\vec{k}}{\sqrt{2\omega_p}}\, e^{i(\omega_p x^0 - \vec{k}\cdot\vec{x})}\,\phi^+(\vec{k})$$

Analogously

$$\phi^-(x) = \frac{1}{(2\pi)^{3/2}}\int dk_0 d^3\vec{k}\, e^{-ikx}\,\delta(k_0^2 - (\vec{k}^2 + m^2))\,\theta(k_0)\,\phi(-k) =$$

$$= \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\omega_p}}\, e^{-i(\omega_p x^0 - \vec{k}\cdot\vec{x})}\,\frac{\phi(-\omega_p, -\vec{k})}{\sqrt{2\omega_p}} = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\omega_p}}\, e^{-i(\omega_p x^0 - \vec{k}\vec{x})}\,\phi^-(\vec{k})$$

where $\phi^-(\vec{k}) \equiv \frac{\phi(-\omega_p, -\vec{k})}{\sqrt{2\omega_p}}$

Conclusion

$$\phi^-(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3\vec{k}}{\sqrt{2\omega_p}}\, e^{-i(\omega_p x^0 - \vec{k}\cdot\vec{x})}\,\phi^-(\vec{k})$$

Therefore:

$$\phi(x) = \phi^+(x) + \phi^-(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3\vec{k}}{\sqrt{2\omega_p}}\left(e^{-i(\omega_p x^0 - \vec{k}\cdot\vec{x})}\,\phi^-(\vec{k}) + e^{i(\omega_p x^0 - \vec{k}\cdot\vec{x})}\,\phi^+(\vec{k})\right)$$

SCALAR FIELD: From momentum representation to dynamical inv.

Dynamical invariants }: functions of fields and their derivatives TIME INDEPENDENT

$$\phi(x) = \phi^+(x) + \phi^-(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{\sqrt{2\omega_p}} e^{i(\omega_p x^0 - \vec{k}\cdot\vec{x})} \phi^+(\vec{k}) + \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 k}{\sqrt{2\omega_p}} e^{-i(\omega_p x^0 - \vec{k}\cdot\vec{x})} \phi^-(\vec{k})$$

Reality condition $\phi(x) = \phi^*(x) \Rightarrow (\phi^-)^* = \phi^+$

Energy-momentum tensor

$$\Theta^{\mu\nu} = \pi^\mu \partial^\nu \phi - g^{\mu\nu} \mathcal{L} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu}\left(\frac{1}{2}(\partial_\sigma \phi \partial^\sigma \phi - m^2\phi^2)\right)$$

Energy-momentum four-vector

$$P^\nu = \int d^3x\, \Theta^{0\nu}$$

$$\boxed{\text{K-G. equation} \quad \partial_\mu \partial^\mu \phi + m^2 \phi = 0}$$

Now

$$P^0 = \int d^3x\, \Theta^{00} = \int d^3x \left(\partial^0\phi \partial^0\phi - \mathcal{L}\right) = \int d^3x \left(\partial^0\phi\partial^0\phi - \frac{1}{2}(\partial^0\phi\partial_0\phi + \partial^i\phi\partial_i\phi - m^2\phi^2)\right) =$$

$\partial_i \phi = -\partial^i\phi$

$$= \frac{1}{2}\int d^3x \left(\partial_0\phi\partial_0\phi + \partial_i\phi\partial_i\phi + m^2\phi^2\right) = \frac{1}{2}\int d^3x\left((\partial_\nu\phi)^2 + m^2\phi^2\right) =$$

$$= \frac{1}{2}\int d^3x \left(\partial_\nu\phi^+(x)\partial_\nu\phi^+(x) + 2\partial_\nu\phi^+(x)\partial_\nu\phi^-(x) + \partial_\nu\phi^-(x)\partial_\nu\phi^-(x) + m^2(\phi^+(x)\phi^+(x) + 2\phi^+\phi^- + \phi^-\phi^-)\right)$$

Obs1: the products of functions of identical frequencies do not contribute i.e

$$\int d^3x\left((\partial_\nu\phi^+)^2 + m^2\phi^{+2}\right) = 0 \quad \text{and so on}$$

Indeed

$$\int d^3x\left((\partial_\nu\phi^+)^2 + m\,\phi^{+2}\right) = \frac{1}{(2\pi)^3}\int d^3k\, d^3k' \frac{1}{\sqrt{2\omega_p}} \frac{1}{\sqrt{2\omega_{p'}}} \phi^+(\vec{k})\phi^+(\vec{k'}) \Big(e^{ik^\mu x_\mu + ik'^\mu x_\mu} m^2$$

$$+ i^2 k_\nu k'_\nu\, e^{ik_\nu x^\nu + ik'_\nu x^\nu}\Big) d^3x =$$

$$= \frac{1}{(2\pi)^3}\int d^3k\, d^3k' \frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_{p'}}} \phi^+(\vec{k})\phi^+(\vec{k'}) e^{i(\omega_p + \omega_{p'})x^0} (m^2 - k_\nu k'_\nu) \underbrace{\int d^3x\, e^{-i(\vec{k}+\vec{k'})\cdot\vec{x}}}_{(2\pi)^3\delta(\vec{k}+\vec{k'})}$$

$$= \int d^3k \frac{1}{2\omega_p}\phi^+(\vec{k})\phi^+(-\vec{k}) e^{2i\omega_p x^0}\left(m^2 - \omega_p^2 + \vec{k}^2\right) = 0$$

$-\omega_p^2 + \vec{k}^2 = -m^2$

$\omega_p = \sqrt{\vec{k}^2 + m^2}$

$\omega_p^2 = \vec{k}^2 + m^2$

and remain

$$P^0 = \int d^3x\left(\partial_\nu\phi^+\partial_\nu\phi^- + m^2\phi^+(x)\phi^-(x)\right) =$$

$$= \frac{1}{(2\pi)^3}\int d^3x \left(\int d^3k\, d^3k'\, \phi^+(\vec{k})\phi^-(\vec{k'})(i^2 k_\nu (-)k'_\nu + m^2) e^{i(k_\mu x^\mu - k'_\mu x^\mu)} \frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_{p'}}}\right.$$

$$= \frac{1}{(2\pi)^3}\int d^3k\, d^3k'\, \phi^+(\vec{k})\phi^-(\vec{k'})\,(m^2 + k_\nu k'_\nu)\frac{e^{i(\omega_p - \omega_{p'})x^0}}{\sqrt{2\omega_p}\sqrt{2\omega_{p'}}} \underbrace{\int d^3x\, e^{i(\vec{k'}-\vec{k})\cdot\vec{x}}}_{(2\pi)^3\delta(\vec{k}-\vec{k'})}$$

Conclusion
$$P^0 = \int d^3k\, \omega_p\, \varphi^+(\vec{k})\varphi^-(\vec{k}) = \frac{1}{2}\int d^3k\, \omega_p \left(\varphi^+(\vec{k})\varphi^-(\vec{k}) + \varphi^-(\vec{k})\varphi^+(\vec{k})\right)$$

Analogously with $\Theta^{0i} = \partial^0\phi\,\partial^i\phi - g^{0i}\mathcal{L} = \partial^0\phi\,\partial^i\phi$

$$P^i = \int d^3x\, \Theta^{0i} = \int d^3x\, (-\partial_0\phi\,\partial_i\phi) = -\int d^3x \left(\left(\partial_0\phi^+ + \partial_0\phi^-\right)\left(\partial_i\phi^+ + \partial_i\phi^-\right)\right) =$$
$$= -\int d^3x \left(\partial_0\phi^+\partial_i\phi^+ + \partial_0\phi^+\partial_i\phi^- + \partial_0\phi^-\partial_i\phi^+ + \partial_0\phi^-\partial_i\phi^-\right) =$$
$$= -\int d^3x \left(\frac{1}{(2\pi)^3}\right)\int d^3k\, d^3k' \frac{i^2}{\sqrt{2\omega_p}\sqrt{2\omega_{p'}}}\Big(\phi^+(\vec{k})\phi^+(\vec{k'})\,\omega_p(-)k'^i e^{i(\omega_p+\omega_{p'})x^0} e^{-i(\vec{k}+\vec{k'})\cdot\vec{x}}$$
$$+ \omega_p k'^i e^{i(\omega_p-\omega_{p'})t} e^{-i(\vec{k}-\vec{k'})\cdot\vec{x}}$$
$$+ \omega_{p'} k^i e^{i(\omega_{p'}-\omega_p)t} e^{i(\vec{k'}-\vec{k})\cdot\vec{x}}$$
$$+(-)\omega_{p'} k'^i e^{-i(\omega_{p'}+\omega_p)t} e^{+i(\vec{k}+\vec{k'})\cdot\vec{x}}$$

$$= -\frac{1}{(2\pi)^3}\int d^3x \int \frac{(-1)\,d^3k}{2\omega_p}\Big(\underbrace{\phi^+(\vec{k})\phi^+(-\vec{k})\,\omega_p\,(+k^i)e^{2i\omega_p x^0}}_{\text{odd fun}} + \phi^+(\vec{k})\phi^-(\vec{k})\,\omega_p k^i$$
$$+ \phi^-(\vec{k})\phi^+(\vec{k})\,\omega_p k^i + \underbrace{\phi^-(\vec{k})\phi^-(-\vec{k})\,\omega_p k^i e^{2i\omega_p x^0}}_{\text{odd function}}\Big)$$

$$= +\int d^3k\, k^i \left(\phi^+(\vec{k})\phi^-(\vec{k})\right) = \frac{1}{2}\int d^3k\; k^i \left(\phi^+(\vec{k})\phi^-(\vec{k}) + \phi^-(\vec{k})\phi^+(\vec{k})\right)$$

Therefore the energy-momentum four-vector is

$$P^\nu = \frac{1}{2}\int d^3k\; k^\nu \left(\phi^+(\vec{k})\phi^-(\vec{k}) + \phi^-(\vec{k})\phi^+(\vec{k})\right)$$

THE SCALAR FIELD

① Definition: $\phi(x)$

Transformation properties $\phi'(x')$

② The Lagrangian: $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2_{(x)}$

③ Euler-Lagrange equations : $\boxed{\partial_\mu \partial^\mu \phi + m^2 \phi = 0}$ Klein-Gordon equation.

or $(\Box^2 - m^2)\phi = 0$ with $\Box^2 = -\partial_\nu \partial^\nu = \nabla^2 - \frac{\partial^2}{\partial t^2}$

④ General Fourier decomposition: $\varphi(x) = \frac{1}{(2\pi)^2} \int d^4k \, e^{ik_\mu x^\mu} \tilde{\varphi}(k)$ $\quad k^\mu = (k^0, \vec{k})$

⑤ Frequency decomposition : $\varphi(x) = \varphi^+(x) + \varphi^-(x)$

$$\begin{cases} \varphi^+(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_p}} e^{i(\omega_p x^0 - \vec{k}\cdot\vec{x})} \varphi^+(\omega_p, \vec{k}) \\ \varphi^-(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_p}} e^{-i(\omega_p x^0 - \vec{k}\cdot\vec{x})} \varphi^-(\omega_p, \vec{k}) \end{cases}$$

$\omega_p = \sqrt{\vec{k}^2 + m^2}$

⑥ The Hamiltonian

$$\Pi_\phi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi$$

$$\mathcal{H} = \Pi_\phi \partial_0 \phi - \mathcal{L} = \Pi_\phi^2 - \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\mathrm{grad}\,\phi)^2 + \frac{1}{2} m^2 \phi^2 =$$

$$= \frac{1}{2}\Pi_\phi^2 + \frac{1}{2}(\mathrm{grad}\,\phi)^2 + \frac{1}{2} m^2 \phi^2$$

⑦ Canonical momentum-energy tensor $\Theta^{\mu\nu} = \Pi^\mu \partial^\nu \phi - g^{\mu\nu}\mathcal{L} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu}\mathcal{L}$

$\Theta^{00} = \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\mathrm{grad}\,\phi)^2 + \frac{1}{2} m^2 \phi^2 \Rightarrow H = \int d^3x \, \Theta^{00}$ → the total Energy of the field

$\Theta^{0i} = \partial^0 \varphi \partial^i \varphi = -\Pi_\phi \partial_i \phi \Rightarrow P^i = \int d^3x \, \Pi_\phi \partial_i \phi$ → Physical momentum carried by the field

8. Invariance at Lorentz transformations

$$\Gamma^{\mu(\rho\sigma)} = g^{\mu\rho} x^{\sigma} - g^{\mu\sigma} x^{\rho}$$

$$G_{\mu} = 0$$

$$W^{\mu(\rho\sigma)} = \mathcal{M}^{\mu(\rho\sigma)}(x) = \theta^{\mu\sigma} x^{\rho} - \theta^{\mu\rho} x^{\sigma}$$

For spatial rotations $\quad M^{ij} = \int d^3x \left(\theta^{0j} x^i - \theta^{0i} x^j \right)$ → orbital angular momentum carried by the field

9. In momentum representation $\quad \varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_p}} \left(e^{-i(\omega_p x^0 - \vec{k}\cdot\vec{x})} \varphi^-(\vec{k}) + e^{i(\omega_p x^0 - \vec{k}\cdot\vec{x})} \varphi^+(\vec{k}) \right)$

$$P^{\mu} = \int d^3k \, k^{\mu} \varphi^+(\vec{k}) \varphi^-(\vec{k}) = \frac{1}{2} \int d^3k \, k^{\mu} \left(\varphi^+(\vec{k}) \varphi^-(\vec{k}) + \varphi^-(\vec{k}) \varphi^+(\vec{k}) \right)$$

THE VECTOR FIELD (MASSIVE)

$V^\mu(x)$; four components which transform as $V'^\mu = \Lambda^\mu{}_\nu V^\mu$ at a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\mu$

$U_\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}$; $(J^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu{}_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)$

α, β → Lorentz indices

$H^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu$; $H^{*\mu\nu} = \partial^\mu V^{*\nu} - \partial^\nu V^{*\mu}$

Lagrangian density · for neutral (real) vector field

$$\mathcal{L}(V^\mu, \partial_\nu V^\mu) = -\frac{1}{4} H^{\mu\nu} H_{\mu\nu} + \frac{m^2}{2} V_\nu V^\nu$$

: for charged vector field

$$\mathcal{L}(V^\mu; \partial_\nu V^\mu, V^{*\mu}, \partial_\nu V^{*\mu}) = -\frac{1}{2} H^{\mu\nu} H^*_{\mu\nu} + m^2 V^\nu V^*_\nu$$

Equations of motion

Observe that
$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu V^\mu)} = -H^{*\nu}{}_\mu \quad ; \quad \frac{\partial \mathcal{L}^*}{\partial(\partial_\nu V^{*\mu})} = -H^\nu{}_\mu$$

$$\frac{\partial \mathcal{L}}{\partial V^\mu} = m^2 V^*_\mu \quad ; \quad \frac{\partial \mathcal{L}}{\partial V^*_\mu} = m^2 V^\mu$$

Then:

$$= -\frac{1}{2}\left(\partial^\nu V_\mu + \partial^\nu V_\mu - \partial_\mu V^\nu - \partial_\mu V^\nu\right) = -\left(\partial^\nu V_\mu - \partial_\mu V^\nu\right) = -H^\nu{}_\mu$$

Then
$$\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu V^\mu)} = -\left(\partial_\nu \partial^\nu V_\mu - \partial_\mu \partial_\nu V^\nu\right)$$

Also
$$\frac{\partial \mathcal{L}}{\partial V^\mu} = \frac{\partial}{\partial V^\mu}\left(\frac{1}{2} m V_\alpha V^\alpha\right) = m^2 V_\mu$$

and the Euler-Lagrange equations $\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu V^\mu)} - \frac{\partial \mathcal{L}}{\partial V^\mu} = 0$ for Proca field are:

$$-\partial_\nu \partial^\nu V_\mu - m^2 V_\mu + \partial_\mu \partial_\nu V^\nu = 0$$

or
$$-\partial_\nu \partial^\nu V^\mu - m^2 V^\mu + \partial^\mu \partial_\nu V^\nu = 0 \qquad (*)$$

Equivalently
$$\begin{cases} -\partial_\nu \partial^\nu V^\mu - m^2 V^\mu = 0 \\ \text{and} \quad \partial_\nu V^\nu = 0 \quad \text{(subsidiary condition)} \end{cases}$$

Observation 1

The subsidiary condition is compatible with the equations of motion

Proof: Take ∂_μ in $(*)$ $\Rightarrow$ $\cancel{-\partial_\nu \partial^\nu \partial_\mu V^\mu} - m^2 \partial_\mu V^\mu + \cancel{\partial_\mu \partial^\mu \partial_\nu V^\nu} = 0 \Rightarrow m^2 \partial_\mu V^\mu = 0 \Rightarrow$

$$\overset{\text{if } m \neq 0}{\Longrightarrow} \partial_\mu V^\mu = 0$$

Observation 2

By employing the Fourier decomposition for each V^μ

$$V^\mu(x) = \frac{1}{(2\pi)^2} \int d^4k \, e^{ik_\mu x^\mu} \tilde{V}^\mu(k)$$

we conclude that $\tilde{V}^\mu(k)$ should verify

$(k^2 - m^2)\tilde{V}^\mu(k) = 0$ if $V^\mu(x)$ satisfy the equations of motion (Proca eq)

Also subsidiary condition imply:

$$\partial_\mu V^\mu = 0 \Rightarrow k_\mu V^\mu = 0 \underset{\text{in rest frame}}{\Longrightarrow} k_0 V^0 - 0 = 0 \Rightarrow V^0 = 0$$

$k^\mu_{RF} = (m, \vec{0})$

So in the rest frame we have 3 non-zero components, one for each projection of spin one

THE MASSIVE VECTOR FIELD (PROCA FIELD)

1. Construction of the action

The field consist of four components V^μ

Under $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ the components of the field tranforms as $V'^\mu = \Lambda^\mu{}_\nu V^\nu$

or for $\delta x^\mu = \omega^\mu{}_\nu x^\nu \Rightarrow \delta V^\mu = V'^\mu(x') - V^\mu(x) = \omega^\mu{}_\nu V^\nu$

Impose on the components V^μ a subsidiary condition, reducing the number of linearly independent components from four to three, one for each of the three possible values of the components of the spin 1 $\rightarrow$ exclusion of a particle of spin zero

$$\partial_\mu V^\mu \equiv \partial U = \frac{\partial V^\mu(x)}{\partial x^\mu} = 0$$

Introduce the antisymmetric tensor

$$H^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu$$

and construct the ~~inv~~ invariant Lagrangian density:

$$\mathcal{L}(V^\mu, \partial_\nu V^\mu) = -\frac{1}{4} H^{\mu\nu} H_{\mu\nu} + \frac{m^2}{2} V_\nu V^\nu$$

2. Euler - Lagrange equations $\rightarrow$ Proca equations

Let us observe that

$$\frac{1}{4} H^{\alpha\beta} H_{\alpha\beta} = \frac{1}{4}(\partial^\alpha V^\beta - \partial^\beta V^\alpha)(\partial_\alpha V_\beta - \partial_\beta V_\alpha) = \frac{1}{4}(\underline{\partial^\alpha V^\beta \partial_\alpha V_\beta} - \underline{\underline{\partial^\alpha V^\beta \partial_\beta V_\alpha}} - \underline{\underline{\partial^\beta V^\alpha \partial_\alpha V_\beta}} + \underline{\partial^\beta V^\alpha \partial_\beta V_\alpha})$$

$$= \frac{1}{2}(\partial^\alpha V^\beta \partial_\alpha V_\beta - \partial^\alpha V^\beta \partial_\beta V_\alpha) = \frac{1}{2}(g^{\alpha\rho} \partial_\rho V^\beta g_{\beta\gamma} \partial_\alpha V^\gamma - g^{\alpha\rho} \partial_\rho V^\beta g_{\alpha\gamma} \partial_\beta V^\gamma)$$

$$= \frac{1}{2}(g^{\alpha\rho} g_{\beta\gamma} \partial_\rho V^\beta \partial_\alpha V^\gamma - g^{\alpha\rho} g_{\alpha\gamma} \partial_\rho V^\beta \partial_\beta V^\gamma)$$

Now

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu V^\mu)} = \frac{\partial}{\partial(\partial_\nu V^\mu)}\left(-\frac{1}{4} H^{\alpha\beta} H_{\alpha\beta}\right) = -\frac{1}{2}\frac{\partial}{\partial(\partial_\nu V^\mu)}\left(g^{\alpha\rho} g_{\beta\gamma} \partial_\rho V^\beta \partial_\alpha V^\gamma - g^{\alpha\rho} g_{\alpha\gamma} \partial_\rho V^\beta \partial_\beta V^\gamma\right) =$$

$$= -\frac{1}{2}\Big(g^{\alpha\rho} g_{\beta\gamma} \delta_{\nu\rho} \delta^{\beta\mu} \partial_\alpha V^\gamma + g^{\alpha\rho} g_{\beta\gamma} \delta_{\nu\alpha} \delta^{\gamma\mu} \partial_\rho V^\beta -$$

$$- g^{\alpha\rho} g_{\alpha\gamma} \delta_{\nu\rho} \delta^{\beta\mu} \partial_\beta V^\gamma - g^{\alpha\rho} g_{\alpha\gamma} \delta_{\nu\beta} \delta^{\gamma\mu} \partial_\rho V^\beta\Big) =$$

$$= -\frac{1}{2}\left(g^{\alpha\nu} g_{\mu\gamma} \partial_\alpha V^\gamma + g^{\nu\rho} g_{\beta\mu} \partial_\rho V^\beta - g^{\alpha\nu} g_{\alpha\gamma} \partial_\mu V^\gamma - g^{\alpha\rho} g_{\alpha\mu} \partial_\rho V^\nu\right) =$$

Dynamical invariants

$$\Theta^{\mu\nu} = - H^{\mu}{}_{\sigma}\,\partial^{\nu} V^{*\sigma} - H^{*\mu}{}_{\sigma}\,\partial^{\nu} V^{\sigma} - g^{\mu\nu}\mathcal{L}$$

$$J^{\nu} = i\left(V^{*\sigma} H^{\nu}{}_{\sigma} - V^{\sigma} H^{*\nu}{}_{\sigma}\right)$$

$$S^{\mu(\rho\sigma)} = V^{*\rho} H^{\sigma\mu} - V^{*\sigma} H^{\rho\mu} + V^{\rho} H^{*\sigma\mu} - V^{\sigma} H^{*\rho\mu}$$

Frequency decomposition

$$V^{\mu}(x) = V^{+\mu}(x) + V^{-\mu}(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\omega_p}}\left(V^{-\mu}(\vec{k})e^{-ikx} + V^{+\mu}(\vec{k})e^{ikx}\right)$$

$$V^{*\mu}(x) = V^{*+\mu}(x) + V^{*-\mu}(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\omega_p}}\left(\overset{*}{V}{}^{-\mu}(\vec{k})e^{-ikx} + \overset{*}{V}{}^{+\mu}(\vec{k})e^{ikx}\right)$$

Observe that $\left(V^{-\mu}(\vec{k})\right)^{*} = \overset{*}{V}{}^{+\mu}$; $\left(V^{+\mu}(\vec{k})\right)^{*} = \overset{*}{V}{}^{-\mu}(\vec{k})$

Dynamical Invariants in terms of frequency coeff.

$$P^{\nu} = -\int d^3k\; k^{\nu}\left[\overset{*}{V}{}^{-}_{\mu}(\vec{k})V^{+\mu}(\vec{k}) + \overset{*}{V}{}^{+}_{\mu}(\vec{k})V^{-\mu}(\vec{k})\right]$$

$$Q = \int d^3k \left[\overset{*}{V}{}^{-}_{\mu}(\vec{k})V^{+\mu}(\vec{k}) - \overset{*}{V}{}^{+}_{\mu}(\vec{k})V^{-\mu}(\vec{k})\right]$$

$$\vec{S} = i\int d^3k \left[\vec{\overset{*}{V}}{}^{+}(\vec{k}) \times \vec{V}^{-}(\vec{k}) - \vec{\overset{*}{V}}{}^{-}(\vec{k}) \times \vec{V}^{+}(\vec{k})\right]$$

Subsidiary conditions

$$\partial^{\mu} V_{\mu} = 0 \Rightarrow \begin{cases} k^{\nu} V^{\pm}_{\nu}(\vec{k}) = 0 \\ k^{\nu} \overset{*}{V}{}^{\pm}_{\nu}(\vec{k}) = 0 \end{cases} \Rightarrow \begin{cases} V^{\pm}_{0}(\vec{k}) = \frac{1}{k_0}\vec{k}\cdot\vec{V}^{\pm}(\vec{k}) \\ \overset{*}{V}{}^{\pm}_{0}(\vec{k}) = \frac{1}{k_0}\vec{k}\cdot\vec{\overset{*}{V}}{}^{\pm}(\vec{k}) \end{cases}$$

Then $$-\overset{*}{V}{}^{\pm}_{\mu} V^{\mp\mu} = \vec{\overset{*}{V}}{}^{\pm}(\vec{k})\cdot\vec{V}^{\mp}(\vec{k}) - \frac{1}{k_0^2}\left(\vec{k}\cdot\vec{\overset{*}{V}}{}^{\pm}(\vec{k})\right)\left(\vec{k}\cdot\vec{V}^{\mp}(\vec{k})\right)$$

The local reference frame with $\vec{e}_1, \vec{e}_2$ → transverse polarization vectors we have

$$\vec{V}^{\pm}(\vec{k}) = \vec{e}_1 a^{\pm}_1(\vec{k}) + \vec{e}_2 a^{\pm}_2(\vec{k}) + \frac{k_0}{m}\frac{\vec{k}}{|\vec{k}|} a^{\pm}_3(\vec{k})$$

and

$$-\overset{*}{V}{}^{\pm}_{\nu}(\vec{k})V^{\mp\nu}(\vec{k}) = \overset{*}{a}{}^{\pm}_{i}(\vec{k})\, a^{\mp}_{i}(\vec{k})$$ and then

$$P^{\nu} = \int d^3k\; k^{\nu}\left[\overset{*}{a}{}^{+}_{i}(\vec{k})\, a^{-}_{i}(\vec{k}) + \overset{*}{a}{}^{-}_{i}(\vec{k})\, a^{+}_{i}(\vec{k})\right]$$

$$Q = \int d^3k \left[\overset{*}{a}{}^{+}_{i}(\vec{k})\, a^{-}_{i}(\vec{k}) - \overset{*}{a}{}^{-}_{i}(\vec{k})\, a^{+}_{i}(\vec{k})\right]$$

THE VECTOR FIELD

Comment 1 → very important in particle physics.

The associated quanta, the vector bosons, play a central role as the mediators of interactions

Examples:
- massless photons → electromagnetic interactions
- $W^{\pm}$ and Z^{0} bosons → weak interaction
- massless gluons → strong interaction
- spin 1 mesons : ρ and ω particles

The vector field consist of four components V^{μ} which under a Lorentz transformation $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ transform with a four-dimensional representation of Lorentz group.

$$U_{\Lambda} = e^{-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}$$

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right)$$ → is indeed antisymmetric to the interchange $\mu \leftrightarrow \nu$

α, β are themselves Lorentz indices and we deal with a 4×4 matrix.

Comment: the variation of field under an infinitesimal Lorentz transf. is

$$\delta V^{\mu}(x) = \omega^{\mu}{}_{\nu} V^{\nu}(x)$$

Proof: Step 1. Let us observe that $\mathcal{J}^{\mu\nu}$ defined above verify the Lorentz group algebra

Indeed, let us observe first that

$$(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta} = g^{\alpha\rho}(\mathcal{J}^{\mu\nu})_{\rho\beta} = g^{\alpha\rho} i(\delta^{\mu}_{\rho}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\rho}) = i\left(g^{\mu\alpha}\delta^{\nu}_{\beta} - g^{\nu\alpha}\delta^{\mu}_{\beta}\right)$$

$$(\mathcal{J}^{\mu\nu})^{\alpha\beta} = g^{\beta\sigma}(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\sigma} = i(g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta})$$

Now

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}]^{\alpha}{}_{\gamma} = (\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta}(\mathcal{J}^{\rho\sigma})^{\beta}{}_{\gamma} - (\mathcal{J}^{\rho\sigma})^{\alpha}{}_{\beta}(\mathcal{J}^{\mu\nu})^{\beta}{}_{\gamma} =$$

$$ii\left(g^{\mu\alpha}\delta^{\nu}_{\beta}-g^{\nu\alpha}\delta^{\mu}_{\beta}\right)\left(g^{\rho\beta}\delta^{\sigma}_{\gamma}-g^{\sigma\beta}\delta^{\rho}_{\gamma}\right)-ii\left(g^{\rho\alpha}\delta^{\sigma}_{\beta}-g^{\sigma\alpha}\delta^{\rho}_{\beta}\right)\left(g^{\mu\beta}\delta^{\nu}_{\gamma}-g^{\nu\beta}\delta^{\mu}_{\gamma}\right)=$$

$$= i\Big(ig^{\mu\alpha}\delta^{\nu}_{\beta}g^{\rho\beta}\delta^{\sigma}_{\gamma} \to ig^{\mu\alpha}g^{\nu\rho}\delta^{\sigma}_{\gamma} \qquad -ig^{\rho\alpha}\delta^{\sigma}_{\beta}g^{\mu\beta}\delta^{\nu}_{\gamma} \to -ig^{\rho\alpha}g^{\mu\sigma}\delta^{\nu}_{\gamma}$$

$$-ig^{\mu\alpha}\delta^{\nu}_{\beta}g^{\sigma\beta}\delta^{\rho}_{\gamma} \to -ig^{\mu\alpha}g^{\nu\sigma}\delta^{\rho}_{\gamma} \qquad +ig^{\rho\alpha}\delta^{\sigma}_{\beta}g^{\nu\beta}\delta^{\mu}_{\gamma} \to ig^{\rho\alpha}g^{\nu\sigma}\delta^{\mu}_{\gamma}$$

$$-ig^{\nu\alpha}\delta^{\mu}_{\beta}g^{\rho\beta}\delta^{\sigma}_{\gamma} \to -ig^{\nu\alpha}g^{\mu\rho}\delta^{\sigma}_{\gamma} \qquad +ig^{\sigma\alpha}\delta^{\rho}_{\beta}g^{\mu\beta}\delta^{\nu}_{\gamma} \to ig^{\sigma\alpha}g^{\mu\rho}\delta^{\nu}_{\gamma}$$

$$+ig^{\nu\alpha}\delta^{\mu}_{\beta}g^{\sigma\beta}\delta^{\rho}_{\gamma} \to ig^{\nu\alpha}g^{\mu\sigma}\delta^{\rho}_{\gamma} \qquad -ig^{\sigma\alpha}\delta^{\rho}_{\beta}g^{\nu\beta}\delta^{\mu}_{\gamma} \to -ig^{\sigma\alpha}g^{\nu\rho}\delta^{\mu}_{\gamma}$$

$$= i\left\{\begin{array}{l} \big(g^{\mu\sigma} i\left(g^{\nu\alpha}\delta^{\rho}_{\gamma}-g^{\rho\alpha}\delta^{\nu}_{\gamma}\right) \\ + g^{\nu\rho} i\left(g^{\mu\alpha}\delta^{\sigma}_{\gamma}-g^{\sigma\alpha}\delta^{\mu}_{\gamma}\right) \\ - g^{\mu\rho} i\left(g^{\nu\alpha}\delta^{\sigma}_{\gamma}-g^{\sigma\alpha}\delta^{\nu}_{\gamma}\right) \\ - g^{\nu\sigma} i\left(g^{\mu\alpha}\delta^{\rho}_{\gamma}-g^{\rho\alpha}\delta^{\mu}_{\gamma}\right)\big) \end{array}\right\} = i\left(g^{\mu\sigma}(J^{\nu\rho})^{\alpha}{}_{\beta}+g^{\nu\rho}(J^{\mu\sigma})^{\alpha}{}_{\beta}-g^{\mu\rho}(J^{\nu\sigma})^{\alpha}{}_{\beta}-g^{\nu\sigma}(J^{\mu\rho})^{\alpha}{}_{\beta}\right)$$

qed

Now

$$V'^{\alpha}=\Lambda^{\alpha}{}_{\beta}V^{\beta}=\left(\delta^{\alpha}_{\beta}-\frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^{\alpha}{}_{\beta}\right)V^{\beta} \Rightarrow \delta V^{\alpha}=V'^{\alpha}-V^{\alpha}=-\frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^{\alpha}{}_{\beta}V^{\beta}$$

$$=-\frac{i}{2}\omega_{\mu\nu}\, i\left(g^{\mu\alpha}\delta^{\nu}_{\beta}-g^{\nu\alpha}\delta^{\mu}_{\beta}\right)V^{\beta}=-\frac{i}{2}\,i\,\omega^{\alpha}{}_{\beta}V^{\beta}+\frac{i}{2}\,i\,\omega_{\beta}{}^{\alpha}V^{\beta}=\omega^{\alpha}{}_{\beta}V^{\beta}$$

or $+\frac{i}{2}i(-)\omega_{\nu\mu}g^{\nu\alpha}\delta^{\mu}_{\beta}$ OK

=

THE MASSIVE VECTOR FIELD (PROCA FIELD)

1. Construction of the action

The field consists of four components V^μ, forming the contravariant four-vector which under a Lorentz transformation $x'^\mu = \Lambda^\mu_{\ \nu} x^\nu$ transforms as

$$V'^\mu = \Lambda^\mu_{\ \nu} V^\nu \quad \text{or for } \delta x^\mu = \omega^\mu_{\ \nu} x^\nu \Rightarrow \delta V^\mu = \omega^\mu_{\ \nu} V^\nu$$

Step 1. Consider a „good" Lagrangian, treating all dynamical variables to form covariant sums i.e.

$$\mathcal{L}_0 = \underbrace{-\frac{1}{2}\partial_\mu V^\nu \partial^\mu V_\nu}_{\text{generalisation from the corresponding expression for scalar field}} + \frac{m^2}{2} V_\nu V^\nu \rightarrow$$

How it was choosen the sign for mass term → agree with KG for space comp

Obs: One may add a term proportional to $\partial_\mu V^\nu \partial_\nu V^\mu$ but this is equivalent to $\partial_\mu V^\mu \partial_\nu V^\nu = \left(\frac{\partial V}{\partial x}\right)^2$

Important: the term connected with the component V_0 enters into such sums with a negative sign and its contribution to the energy turns out to be negative. The way out of this difficulty consists of imposing on V^μ the invariant subsidiary condition $\partial_\mu V^\mu = \frac{\partial V^\mu(x)}{\partial x^\mu} \equiv \partial U = 0$

Bogol. 29

⇒ reduce the number of linearly independent components from four to three

⇒ will guarantees that the energy of the vector field is positive definite

This condition is equivalent to the exclusion of a particle of spin zero (which in this formulation would lead to a negative energy)

Step 2. Introduce the antisymmetric tensor field

$$H^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu$$

and construct the Lagrangian density

$$\mathcal{L}(V^\mu, \partial_\nu V^\mu) = -\frac{1}{4} H^{\mu\nu} H_{\mu\nu} + \frac{m^2}{2} V_\nu V^\nu$$

(by appropriate modification of the above Lagrangian $\mathcal{L}_0$ adding a term $\partial_\mu V^\nu \partial_\nu V^\mu$ which reduces to a four-divergence)

2 Euler Lagrange equations (Proca equations)

Let us observe that:

$$\frac{1}{4}H^{\alpha\beta}H_{\alpha\beta} = \frac{1}{4}\left(\partial^\alpha V^\beta - \partial^\beta V^\alpha\right)\left(\partial_\alpha V_\beta - \partial_\beta V_\alpha\right) = \frac{1}{4}\left(\underline{\partial^\alpha V^\beta \partial_\alpha V_\beta} - \partial^\alpha V^\beta \partial_\beta V_\alpha - \partial^\beta V^\alpha \partial_\alpha V_\beta + \underline{\partial^\beta V^\alpha \partial_\beta V_\alpha}\right) =$$

$$= \frac{1}{2}\left(\partial^\alpha V^\beta \partial_\alpha V_\beta - \partial^\alpha V^\beta \partial_\beta V_\alpha\right) = \frac{1}{2}\left(g^{\alpha\rho}\partial_\rho V^\beta g_{\beta\gamma}\partial_\alpha V^\gamma - g^{\alpha\rho}\partial_\rho V^\beta g_{\alpha\gamma}\partial_\beta V^\gamma\right) =$$

$$= \frac{1}{2}\left(g^{\alpha\rho} g_{\beta\gamma}\partial_\rho V^\beta \partial_\alpha V^\gamma - g^{\alpha\rho} g_{\alpha\gamma}\partial_\rho V^\beta \partial_\beta V^\gamma\right)$$

Now

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu V^\mu)} = \frac{\partial}{\partial(\partial_\nu V^\mu)}\left(-\frac{1}{4}H^{\alpha\beta}H_{\alpha\beta}\right) = -\frac{1}{2}\frac{\partial}{\partial(\partial_\nu V^\mu)}\left(g^{\alpha\rho} g_{\beta\gamma}\partial_\rho V^\beta \partial_\alpha V^\gamma - g^{\alpha\rho} g_{\alpha\gamma}\partial_\rho V^\beta \partial_\beta V^\gamma\right)$$

$$= -\frac{1}{2}\left(g^{\alpha\rho} g_{\beta\gamma}\delta_{\nu\rho}\delta^{\beta\mu}\partial_\alpha V^\gamma + g^{\alpha\rho} g_{\beta\gamma}\delta_{\nu\alpha}\delta^{\gamma\mu}\partial_\rho V^\beta - g^{\alpha\rho} g_{\alpha\gamma}\delta_{\nu\rho}\delta^{\beta\mu}\partial_\beta V^\gamma - g^{\alpha\rho} g_{\alpha\gamma}\delta_{\nu\beta}\delta^{\mu\gamma}\partial_\rho V^\beta\right)$$

$$= -\frac{1}{2}\left(g^{\alpha\nu} g_{\mu\gamma}\partial_\alpha V^\gamma + g^{\nu\rho} g_{\beta\mu}\partial_\rho V^\beta - g^{\alpha\nu} g_{\alpha\gamma}\partial_\mu V^\gamma - g^{\alpha\rho} g_{\alpha\mu}\partial_\rho V^\nu\right)$$

$$= -\frac{1}{2}\left(\partial^\nu V_\mu + \partial^\nu V_\mu - \partial_\mu V^\nu - \partial_\mu V^\nu\right) = -\left(\partial^\nu V_\mu - \partial_\mu V^\nu\right) = -H^\nu{}_\mu$$

Then $$\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu V^\mu)} = -\left(\partial_\nu \partial^\nu V_\mu - \partial_\mu \partial_\nu V^\nu\right)$$

Also observe that $$\frac{\partial \mathcal{L}}{\partial V^\mu} = \frac{\partial}{\partial V^\mu}\left(\frac{1}{2}m V_\alpha V^\alpha\right) = m^2 V_\mu$$

The Euler-Lagrange equations $\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu V^\mu)} - \frac{\partial \mathcal{L}}{\partial V^\mu} = 0$ for Proca field are

$$-\partial_\nu \partial^\nu V_\mu - m^2 V_\mu + \partial_\mu \partial_\nu V^\nu = 0 \quad \text{or}$$

$$\boxed{-\partial_\nu \partial^\nu V^\mu - m^2 V^\mu + \partial^\mu(\partial_\nu V^\nu) = 0}$$

Equivalently $$\boxed{\begin{array}{l} -\partial_\nu \partial^\nu V^\mu - m^2 V^\mu = 0 \quad \text{and} \\ \partial^\nu V_\nu = 0 \quad \text{(subsidiary condition)} \end{array}}$$ ← each V^μ verify Klein-Gordon eq.

Observation 1 The subsidiary condition is compatible with the equations of motion.

Proof Take ∂_μ on Proca eq $\Rightarrow$ $\cancel{-\partial_\nu \partial^\nu \partial_\mu V^\mu} - m^2 \partial_\mu V^\mu + \cancel{\partial_\mu \partial^\mu \partial_\nu V^\nu} = 0$

$$\Rightarrow \partial_\mu V^\mu = 0 \quad \text{q.e.d.}$$

Observation 2 By employing the Fourier decomposition for each V^μ i.e. $V^\mu(x) = \frac{1}{(2\pi)^2}\int d^4k\, e^{ik_\mu x^\mu}\tilde{V}^\mu(k)$ we conclude that $\tilde{V}^\mu(k)$ should verify $(k^2 - m^2)\tilde{V}^\mu(k) = 0$ if V^μ satisfy eq of motion

$$\partial_\mu V^\mu = 0 \Rightarrow k_\mu V^\mu = 0 \xrightarrow[\text{frame}]{\text{in rest}} k_0 V^0 - 0 = 0 \Rightarrow V^0 = 0$$

So in the rest frame we have 3 non-zero components (one for

Obs ... Eq. Proca is not gauge invariant $k^\mu = (m, \vec{0})$

Momentum representation. Positive and negative frequency decomposition (5V)

A procedure similar to that encountered for the scalar field leads us to the decomposition

$$V_\mu(x) = V_\mu^+(x) + V_\mu^-(x)$$

and for each V_μ

$$\begin{cases} V_\mu^+(x) = \dfrac{1}{(2\pi)^{3/2}} \displaystyle\int \dfrac{d^3\vec{k}}{\sqrt{2\omega_p}}\, e^{i(\omega_p x^0 - \vec{k}\cdot\vec{x})}\, V_\mu^+(\vec{k}) \\ V_\mu^-(x) = \dfrac{1}{(2\pi)^{3/2}} \displaystyle\int \dfrac{d^3\vec{k}}{\sqrt{2\omega_p}}\, e^{-i(\omega_p x^0 - \vec{k}\cdot\vec{x})}\, V_\mu^-(\vec{k}) \end{cases} \qquad \omega_p = \sqrt{\vec{k}^2 + m^2}$$

So the momentum representation of the vector field is

$$V^\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{\sqrt{2\omega_p}} \left(V^{-\mu}(\vec{k})\, e^{-i(\omega_p x^0 - \vec{k}\cdot\vec{x})} + V^{+\mu}(\vec{k})\, e^{i(\omega_p x^0 - \vec{k}\cdot\vec{x})} \right)$$

If the vector field V^μ is complex, an analogous expansion occurs for $V^{*\mu}(x)$ i.e.

$$V^{*\mu}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{\sqrt{2\omega_p}} \left(V^{*-\mu}(\vec{k})\, e^{-i(\omega_p x^0 - \vec{k}\cdot\vec{x})} + V^{*+\mu}(\vec{k})\, e^{i(\omega_p x^0 - \vec{k}\cdot\vec{x})} \right)$$

Let us explore the consequences of subsidiary conditions:

$\partial_\mu V^\mu = 0 \Rightarrow k^\nu V_\nu^\pm(\vec{k}) = 0$ i.e. the four components $V_\nu(\vec{k})$ are no longer independent and the components $V_0^\pm(\vec{k})$ can be expressed through the remaining ones:

$$V_0^\pm(\vec{k}) = -\frac{1}{k_0} k^i V_i^\pm(\vec{k}) = \frac{1}{k_0} \vec{k}\cdot\vec{V}^\pm(\vec{k})$$

Using this relation let us observe that $-V_\nu^\pm(\vec{k}) V^{\nu\mp}(\vec{k})$ becomes:

$$-V_\nu^\pm(\vec{k}) V^{\nu\mp}(\vec{k}) = \vec{V}^\pm(\vec{k})\,\vec{V}^\mp(\vec{k}) - V_0^\pm(\vec{k}) V^\mp(\vec{k}) = \vec{V}^\pm(\vec{k})\cdot\vec{V}^\mp(\vec{k}) - \frac{1}{k_0^2}\left(\vec{k}\cdot\vec{V}^\pm(\vec{k})\right)\left(\vec{k}\cdot\vec{V}^\mp(\vec{k})\right)$$

Local reference frame: Introduce two units vectors $\vec{e}_1$ and $\vec{e}_2$ orthogonal to the wave vector $\vec{k}$ and to each other as well as $\vec{e}_3 = \frac{\vec{k}}{|\vec{k}|}$ along the wave vector.

$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$; $i,j = \overline{1,3}$; $\vec{e}_1, \vec{e}_2 \to$ transverse polarisation vectors.

Then the spatial part of $V^\mu(\vec{k})$ i.e. $\vec{V}(\vec{k})$ can be decomposed as follows:

$$\vec{V}(\vec{k}) = \underbrace{\vec{e}_1\, a_1(\vec{k}) + \vec{e}_2\, a_2(\vec{k})}_{\text{transverse components}} + \underbrace{\frac{k_0}{m}\frac{\vec{k}}{|\vec{k}|}\, a_3(\vec{k})}_{\text{longitudinal component}}$$

Therefore

$$-V_\nu^\pm(\vec{k}) V^{\mp\nu}(\vec{k}) = \vec{V}^\pm(\vec{k})\cdot\vec{V}^\mp(\vec{k}) - \frac{1}{k_0^2}\left(\vec{k}\cdot\vec{V}^\pm(\vec{k})\right)\left(\vec{k}\cdot\vec{V}^\mp(\vec{k})\right) =$$

$$= a_1^\pm(\vec{k}) a_1^\mp(\vec{k}) + a_2^\pm(\vec{k}) a_2^\mp(\vec{k}) + \frac{k_0^2}{m^2} a_3^\pm(\vec{k}) a_3^\mp(\vec{k}) - \frac{1}{k_0^2}\frac{k_0^2}{m^2}\vec{k}^2 a_3^\pm(\vec{k}) a_3^\mp(\vec{k}) = a_i^\pm(\vec{k}) a_i^\mp(\vec{k})$$

and the product $V^{\pm}_{\nu}(k) N^{\mp\nu}(k)$ is consequently diagonalized by (6V)
this linear substitution (see also Pauli RMP 13 (1940) pag 216): This is simply the transformation to the principal axis.

On polarization vectors

$\varepsilon_\mu(\vec{k},\lambda) \to$ polarization vectors (four-vectors): play a similar role as the unit spinors u & v in plane-wave decomp of the Dirac field

↳ counts the different polarization states

$\varepsilon^\mu \to$ three space-like ; one time-like

Simplification → polarization vectors are defined with respect to the direction of the momentum vector

Let us demand that the polarization vectors form a four-dimensional orthonorm system satisfying

$$\varepsilon_\mu(\vec{k},\lambda)\,\varepsilon^\mu(\vec{k},\lambda') = g_{\lambda\lambda'} \qquad (6.62)$$

The Massive vector field case

Choose the system of reference ; here the plane wave has momentum $\vec{k}$

Step 1: Choose two space-like transverse polarization vectors

$$\varepsilon(\vec{k},1) = (0, \vec{\varepsilon}(\vec{k},1))$$
$$\varepsilon(\vec{k},2) = (0, \vec{\varepsilon}(\vec{k},2))$$

by imposing the conditions
$$\begin{cases} \vec{\varepsilon}(\vec{k},1)\cdot\vec{k} = \vec{\varepsilon}(\vec{k},2)\cdot\vec{k} = 0 \\ \vec{\varepsilon}(\vec{k},i)\cdot\vec{\varepsilon}(\vec{k},j) = \delta_{ij} \end{cases}$$

Step 2: The third polarization vector $(\lambda=3)$ is constructed such that its spatial component points in the direction of the momentum $\vec{k}$ while being normalized according to (6.62)

Adopt the further condition that the four-vector $\varepsilon_\mu(\vec{k},3)$ is orthogonal to the momentum four-vector i.e.

$$k^\mu \varepsilon_\mu(\vec{k},3) = 0 \qquad (6.66)$$

So the components of longitudinal polarization vector are:

$$\varepsilon(\vec{k},3) = \left(\frac{|\vec{k}|}{m}, \frac{\vec{k}}{|\vec{k}|}\frac{k_0}{m}\right) \to \text{This verify both} \begin{cases} k^\mu \varepsilon_\mu(\vec{k},3) = 0 \\ \varepsilon_\mu(\vec{k},3)\,\varepsilon^\mu(\vec{k},3) = -1 \end{cases}$$

along $\vec{k}$

not well defined in the limit $m\to 0$ ⇒ would be not possible to construct a vector that is transverse in 4-dimen

$$k^0\frac{|\vec{k}|}{m} - \frac{k^0}{m}\frac{\vec{k}^2}{|\vec{k}|} = 0$$

$$\frac{\vec{k}^2}{m^2} - \frac{k_0^2}{m^2}\vec{k}^2\frac{1}{\vec{k}^2} = \frac{\vec{k}^2 - k_0^2}{m^2} = \frac{-m^2}{m^2} = -1$$

Step 3: Complete the vector basis in Minkowski space : a time-like polarization vector with $\lambda = 0$ → simply use the momentum vector k i.e.

$$\varepsilon(\vec{k},0) = \frac{1}{m}k$$

→ ensures the normalization. $\frac{1}{m^2}k_\mu k^\mu = \frac{k_0^2 - \vec{k}^2}{m^2} = 1$

$$k\cdot\varepsilon(\vec{k},1) = k\cdot\varepsilon(\vec{k},2) = k\cdot\varepsilon(\vec{k},3) = 0$$ → i.e. each of the three vectors $\varepsilon(\vec{k},\lambda)$ satisfy only the subsidiary condition

$$k\cdot\varepsilon(\vec{k},0) = m$$

Observation The four polarization vectors satisfy a completeness relation

For Massless Vector field

Step 1: As in the massive case introduce first two transverse polarization vectors, with $\lambda = 1, 2$ which refer to a fixed Lorentz system

Obs: However now momentum quadrivector k can no longer be used as a basis vector. k cannot be normalised to 1 since $k^2 = 0$. In addition for $m = 0$ the longitudinal polarization vector $\left(\frac{|\vec{k}|}{m}, \frac{\vec{k}}{|\vec{k}|}\frac{k_0}{m}\right)$ is not defined

Conclusion: in the massless case it is impossible to construct a third polarization vector which is normalizable and at the same time transverse (in four dimensions)

Step 2: Arbitrarily define a time-like unit vector which in the chosen special Lorentz frame simply is given by:

$$n = (1, 0, 0, 0) \qquad n^2 = 1$$

and the longitudinal polarization vector can then be written in covariant form as

$$\epsilon(\vec{k}, 3) = \frac{k - n(k \cdot n)}{[(k \cdot n)^2 - k^2]^{1/2}}$$

Observe that: $(\epsilon(\vec{k}, 3))(\epsilon(\vec{k}, 3)) = \frac{k^2 - (k \cdot n)^2 - (k \cdot n)^2 + n^2 (k \cdot n)^2}{(k n)^2 - k^2} = -1$

i.e. correct normalisation (-1)

$$n \cdot \epsilon(\vec{k}, 3) = \frac{(n k) - n^2 (n\, k)}{[(k \cdot n)^2 - k^2]^{1/2}} = 0$$

In the special Lorentz frame it reduces to $\epsilon(\vec{k}, 3) = \left(0, \frac{\vec{k}}{|\vec{k}|}\right)$ $\longrightarrow$ orthogonal to the transverse vectors

Step 3: Complete with the fourth member of our vector basis using n as a time like polarization vector i.e

$$\epsilon(\vec{k}, 0) = n$$

$$\epsilon(\vec{k}, 0) \cdot \epsilon(\vec{k}, 0) = n^2 = 1 \quad \text{and} \quad \epsilon(\vec{k}, 0) \cdot \epsilon(\vec{k}, i) = 0$$

Observation: the polarisation basis in an arbitrary inertial frame can be obtained via a Lorentz transformation, giving a more complicated appearance to the vectors $\epsilon^\mu(\vec{k}, \lambda)$ which in general will consist of a mixture of space and time components. Now

$$k \cdot \epsilon(\vec{k}, 1) = k \cdot \epsilon(\vec{k}, 2) = 0 \; ; \quad k \cdot \epsilon(\vec{k}, 0) = -k \cdot \epsilon(\vec{k}, 3) = k \cdot n$$

valid in any frame of reference.

General comment: The three polarisation states for vector massive case on their own do not form a complete set (in math sense), but will contain an extra term (obtained from general completion relation)

$$\sum_{\lambda=1}^{3} \varepsilon_\mu(\vec{k}, \lambda)\, \varepsilon_\nu(\vec{k}, \lambda) = -\left(g_{\mu\nu} - \frac{1}{m^2} k_\mu k_\nu\right) \qquad \text{(massive case)}$$

$$\sum^{2} \varepsilon_\mu(\vec{k}, \lambda)\, \varepsilon_\nu(\vec{k}, \lambda) = -g_{\mu\nu} - \frac{k_\mu k_\nu}{(k \cdot n)^2} + \frac{k_\mu n_\nu + k_\nu n_\mu}{k \cdot n} \qquad \text{(For free photons } k^2 = 0\text{)}$$

Dynamical invariants for massive vector field

$$\mathcal{L} = -\frac{1}{4} H_{\mu\nu} H^{\mu\nu} + \frac{1}{2} m^2 V_\nu V^\nu$$ for a real vector field

$$\mathcal{L} = -\frac{1}{2} H^*_{\mu\nu} H^{\mu\nu} + m^2 V^*_\nu V^\nu$$ for a complex vector field

$$H^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu \quad ; \quad H^{*\mu\nu} = \partial^\mu V^{*\nu} - \partial^\nu V^{*\mu}$$

Remainder $\quad \Theta^{\mu\nu} = \sum_r \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi_r}{\partial x^\mu}\right)} \frac{\partial \varphi_r}{\partial x_\nu} - g^{\mu\nu}\mathcal{L}$

In our case:

For real vector field

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu V^\rho)} \partial^\nu V^\rho - g^{\mu\nu}\left(-\frac{1}{4} H_{\sigma\rho} H^{\sigma\rho} + \frac{1}{2} m^2 V_\rho V^\rho\right) = \frac{1}{4} H_{\sigma\rho} H^{\sigma\rho} g^{\mu\nu} - g^{\mu\nu}\frac{1}{2} m^2 V^\rho V_\rho$$

$$- H^\mu{}_\rho \partial^\nu V^\rho = \frac{1}{4} g^{\mu\nu} H_{\sigma\rho} H^{\sigma\rho} - g^{\mu\nu}\frac{1}{2} m^2 V^\rho V_\rho - H^\mu{}_\rho \partial^\nu V^\rho$$

$$= \frac{1}{4} g^{\mu\nu} H_{\sigma\rho} H^{\sigma\rho} - H^{\mu\rho}\partial^\nu V_\rho - \frac{1}{2} m^2 g^{\mu\nu} V_\rho V^\rho$$

For complex vector field $\quad$ add $\partial_\sigma(H^{\mu\sigma} A^\nu)$ symmetrization $\rightarrow$ $T^{\mu\nu} = \frac{1}{4} g^{\mu\nu} H_{\sigma\rho} H^{\sigma\rho} + H^{\mu\sigma} H^\nu{}_\sigma - \frac{1}{2} m^2 g^{\mu\nu} V_\rho V^\rho + m^2 V^\mu V^\nu$

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu V^\rho)} \partial^\nu V^\rho + \frac{\partial \mathcal{L}}{\partial(\partial_\mu V^{*\rho})} \partial^\nu V^{*\rho} - g^{\mu\nu}\left(-\frac{1}{2} H^*_{\mu\nu} H^{\mu\nu} + m^2 V^*_\nu V^\nu\right) =$$

$$= \frac{1}{2} g^{\mu\nu} H^*_{\mu\nu} H^{\mu\nu} - g^{\mu\nu} m^2 V^*_\nu V^\nu - H^{*\mu}{}_\rho \partial^\nu V^\rho - H^\mu{}_\rho \partial^\nu V^{*\rho} =$$

$$T^{00} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \frac{1}{2} m^2 (V^{0\,2} + \vec{V}^2)$$

$$T^{0i} = \vec{E}\times\vec{B} - m^2 V^0 V^i$$

since

$$\left(\frac{\partial}{\partial(\partial_\mu V^{*\rho})} H^*_{\alpha\beta} H^{\alpha\beta}\right)\partial^\nu V^{*\rho} = \frac{\partial}{\partial(\partial_\mu V^{*\rho})}\left(\partial_\alpha V^*_\beta - \partial_\beta V^*_\alpha\right) H^{\alpha\beta} \partial^\nu V^{*\rho} =$$

$$= \frac{\partial}{\partial(\partial_\mu V^{*\rho})}\left(g_{\beta\gamma}\partial_\alpha V^{*\gamma} - g_{\alpha\gamma}\partial_\beta V^{*\gamma}\right) H^{\alpha\beta}\partial^\nu V^{*\rho} = \left(g_{\beta\gamma}\delta^\mu_\alpha \delta^\gamma_\rho - g_{\alpha\gamma}\delta^\mu_\beta \delta^\gamma_\rho\right) H^{\alpha\beta}\partial^\nu V^{*\rho} =$$

$$= g_{\beta\rho}\delta^\mu_\alpha H^{\alpha\beta}\partial^\nu V^{*\rho} - g_{\alpha\rho}\delta^\mu_\beta H^{\alpha\beta}\partial^\nu V^{*\rho} = g_{\beta\rho} H^{\mu\beta}\partial^\nu V^{*\rho} - g_{\alpha\rho} H^{\alpha\mu}\partial^\nu V^{*\rho} =$$

$$= H^\mu{}_\rho \partial^\nu V^{*\rho} + H^\mu{}_\rho \partial^\nu V^{*\rho} = \underline{2 H^\mu{}_\rho \partial^\nu V^{*\rho}}$$

But.

$$\frac{i}{2}\int d^3x\, \Psi^+\partial^\nu\Psi = \frac{i}{2}\frac{1}{(2\pi)^3}\int\frac{d^3k\,d^3k'}{2\sqrt{\omega_p\omega_{p'}}}\sum_{s,s'}\Big((b^+_{s'}(\vec{k}'))^* a^-_s(\vec{k})\, v^+_{s'}(\vec{k}')u_s(\vec{k})(-ik^\nu)\int d^3x\, e^{+i(\vec{k}'+\vec{k})\cdot\vec{x}}\, e^{-i(\omega_{p'}+\omega_p)x^0} +$$

$$(b^+_{s'}(\vec{k}'))^* b^+_s(\vec{k})\, v^+_{s'}(\vec{k}')v_s(\vec{k})(ik^\nu)\int d^3x\, e^{i(\vec{k}'-\vec{k})\cdot\vec{x}}\, e^{-i(\omega_{p'}-\omega_p)x^0}$$

$$+(a^-_{s'}(\vec{k}'))^* a^-_s(\vec{k})\, u^+_{s'}(\vec{k}')u_s(\vec{k})(-ik^\nu)\int d^3x\, e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}}\, e^{+i(\omega_{p'}-\omega_p)x^0}$$

$$+(a^-_{s'}(\vec{k}'))^* b^+_s(\vec{k})\, u^+_{s'}(\vec{k}')v_s(\vec{k})(ik^\nu)\int d^3x\, e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}}\, e^{+i(\omega_{p'}+\omega_p)x^0}$$

$$= \frac{i}{2}\frac{1}{(2\pi)^3}(2\pi)^3\int\frac{d^3k\,d^3k'}{2\sqrt{\omega_p\omega_{p'}}}\sum_{s,s'}\Big((-ik^\nu)\,\delta(\vec{k}'+\vec{k})\,(b^+_{s'}(\vec{k}'))^* a^-_s(\vec{k})\, v^+_{s'}(\vec{k}')u_s(\vec{k})\, e^{-i(\omega_{p'}+\omega_p)x^0}$$

$$(+ik^\nu)\,\delta(\vec{k}'-\vec{k})\,(b^+_{s'}(\vec{k}'))^* b^+_s(\vec{k})\, v^+_{s'}(\vec{k}')v_s(\vec{k})\, e^{-i(\omega_{p'}-\omega_p)x^0}$$

$$(-ik^\nu)\,\delta(\vec{k}'-\vec{k})\,(a^-_{s'}(\vec{k}'))^* a^-_s(\vec{k})\, u^+_{s'}(\vec{k}')u_s(\vec{k})\, e^{i(\omega_{p'}-\omega_p)x^0}$$

$$(+ik^\nu)\,\delta(\vec{k}'+\vec{k})\,(a^-_{s'}(\vec{k}'))^* b^+_s(\vec{k})\, u^+_{s'}(\vec{k}')v_s(\vec{k})\, e^{i(\omega_{p'}+\omega_p)x^0} =$$

$$= -\frac{i^2}{2}\int\frac{d^3k}{2\omega_p}\sum_{s,s'}\Big(k^\nu (b^+_{s'}(-\vec{k}))^* a^-_s(\vec{k})\, v^+_{s'}(-\vec{k})u_s(\vec{k})\, e^{-2i\omega_p x^0}$$

$$-k^\nu (b^+_{s'}(\vec{k}))^* b^+_s(\vec{k})\, v^+_{s'}(\vec{k})v_s(\vec{k})$$

$$+k^\nu (a^-_{s'}(+\vec{k}))^* (a^-_s(\vec{k}))\, u^+_{s'}(\vec{k})u_s(\vec{k})$$

$$-k^\nu (a^-_{s'}(-\vec{k}))^* b^+_s(\vec{k})\, u^+_{s'}(-\vec{k})v_s(\vec{k})\, e^{2i\omega_p x^0} =$$

$$= \frac{1}{2}\int\frac{d^3k}{\cancel{2\omega_p}}\sum_s k^\nu\Big((a^-_s(\vec{k}))^*(a^-_s(\vec{k})) - (b^+_s(\vec{k}))^*(b^+_s(\vec{k}))\Big)\cdot\cancel{2\omega_p}$$

$$= \frac{1}{2}\int d^3k\sum_s k^\nu\Big((a^-_s(\vec{k}))^* a^-_s(\vec{k}) - (b^+_s(\vec{k}))^*(b^+_s(\vec{k}))\Big)$$

Summing up the two results we obtain that for Dirac field.

$$P^\nu = \int d^3k\sum_s k^\nu\Big((a^-_s(\vec{k}))^* a^-_s(\vec{k}) - (b^+_s(\vec{k}))^*(b^+_s(\vec{k}))\Big)$$

CLASSICAL DIRAC FIELD

Let be $\psi(x)$ the Dirac field, a four-components spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \quad ; \quad \psi^\dagger = \begin{pmatrix} \psi_1^* \\ \psi_2^* \\ \psi_3^* \\ \psi_4^* \end{pmatrix}^T = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$$

$$\bar{\psi} = \psi^\dagger \gamma^0 \rightarrow$$

① The Lagrangian density:

$$\mathcal{L}(x) = \frac{i}{2}\left(\bar{\psi}\gamma^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi})\gamma^\mu \psi\right) - m\bar{\psi}\psi$$

→ this construction leads to a real Lagrangian

but only the part $i\bar{\psi}(\gamma^\mu\partial_\mu - m)\psi$ leads to the same equations of motion

~~Here~~ $\bar{\psi}$ and ψ are treated as independent fields since $\psi_\alpha(x)$ is complex ($\alpha = \overline{1,4}$)

② Euler-Lagrange equations
For each component ϕ^α of the field we have $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)}\right) - \frac{\partial \mathcal{L}}{\partial \phi^\alpha} = 0$

In our case $\phi^\alpha = \psi_\alpha(x)$ or $\phi^\alpha = \bar{\psi}_\alpha(x)$. Then

$$\mathcal{L}(x) = \frac{i}{2}\left(\bar{\psi}_\alpha (\gamma^\mu \partial_\mu)_{\alpha\beta}\psi_\beta - (\partial_\mu \bar{\psi})_\alpha \gamma^\mu_{\alpha\beta}\psi_\beta\right) - m\bar{\psi}_\alpha \psi_\alpha \quad \checkmark$$

and

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_\gamma} = \frac{i}{2}\delta_{\alpha\gamma}(\gamma^\mu\partial_\mu)_{\alpha\beta}\psi_\beta - m\delta_{\alpha\gamma}\psi_\alpha = \frac{i}{2}(\gamma^\mu\partial_\mu)_{\gamma\beta}\psi_\beta - m\psi_\gamma$$

or formally $\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \frac{i}{2}\gamma^\mu\partial_\mu \psi - m\psi$.

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_\gamma)} = \frac{i}{2}(-\delta_{\alpha\gamma})\gamma^\mu_{\alpha\beta}\psi_\beta = -\frac{i}{2}\gamma^\mu_{\gamma\beta}\psi_\beta$$

or formally $\frac{\partial}{\partial(\partial_\mu \bar{\psi})} = -\frac{i}{2}\gamma^\mu \psi$ and therefore

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}\right) = -\frac{i}{2}\gamma^\mu\partial_\mu \psi$$

From these relations the Euler-Lagrange equations are: $-\frac{i}{2}\gamma^\mu\partial_\mu\psi - \frac{i}{2}\gamma^\mu\partial_\mu\psi + m\psi = 0$ or equivalently

$$\boxed{i\gamma^\mu\partial_\mu \psi(x) - m\psi(x) = 0} \; ; \; \boxed{i(\partial_\mu \bar{\psi}(x))\gamma^\mu + m\bar{\psi}(x) = 0}$$

3 The Hamiltonian

Let us observe that the momentum canonically conjugated to ψ is

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial t}\right)} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \frac{i}{2}\bar{\psi}\gamma^0 = \frac{i}{2}\psi^+\gamma^0\gamma^0 = \frac{i}{2}\psi^+ \equiv \Pi_\psi$$

while the momentum canonically conjugated to $\bar{\psi}$ is

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \bar{\psi}}{\partial t}\right)} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} = -\frac{i}{2}\gamma^0\psi = \Pi_{\bar{\psi}}$$

Then the Legendre transformation provides the Hamiltonian density:

$$\mathcal{H}(x) = \Pi_\psi \frac{\partial \psi}{\partial t} + \frac{\partial \bar{\psi}}{\partial t}\Pi_{\bar{\psi}} - \mathcal{L}(x) = \frac{i}{2}\psi^+\frac{\partial \psi}{\partial t} - \frac{i}{2}\frac{\partial \psi^+}{\partial t}\underbrace{\gamma^0\gamma^0}_{I}\psi - \frac{i}{2}\left(\bar{\psi}\gamma^\mu\partial_\mu\psi - (\partial_\mu\bar{\psi})\gamma^\mu\psi\right) + m\bar{\psi}\psi =$$

$$= \frac{i}{2}\psi^+\frac{\partial \psi}{\partial t} - \cancel{\frac{i}{2}\frac{\partial \psi^+}{\partial t}\psi} - \frac{i}{2}\psi^+\gamma^0\gamma^0\frac{\partial \psi}{\partial t} + \cancel{\frac{i}{2}\frac{\partial \psi^+}{\partial t}\gamma^0\gamma^0\psi} - \frac{i}{2}\left(\bar{\psi}\gamma^i\partial_i\psi - (\partial_i\bar{\psi})\gamma^i\psi\right) + m\bar{\psi}\psi =$$

$$= -\frac{i}{2}\left(\psi^+\gamma^0\gamma^i\partial_i\psi - (\partial_i\psi^+)\gamma^0\gamma^i\psi\right) + m\psi^+\gamma^0\psi =$$

$$= -i\left(\psi^+\alpha_i\partial_i\psi\right) + m\psi^+\gamma^0\psi + \frac{i}{2}\partial_i\left(\psi^+\alpha_i\psi\right)$$

The total Hamiltonian is

$$H = \int d^3x\, \psi^+\left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi + \frac{i}{2}\int d^3x\, \underbrace{\partial_i\left(\psi^+\alpha_i\psi\right)}_{\mathrm{div}\,\vec{j}} =$$

$$\Rightarrow \quad H = \int d^3x\, \psi^+\left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi + \frac{i}{2}\underbrace{\oint_{\Sigma_\infty}\vec{j}\cdot\vec{m}\,da}_{0}$$

(at infinity the current $\vec{j} \to 0$)

In conclusion

$$\boxed{H = \int d^3x\, \psi^+\left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi}$$

4. Dynamical invariants for Dirac field

Noether currents to space-time translation:

$$\mathcal{W}^{\mu\nu}(x) = \Theta^{\mu\nu}(x) \quad \rightarrow \text{ the stress-energy tensor}$$

$$\Theta^{\mu\nu}(x) = \sum_r \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi_r}{\partial x^\mu}\right)} \frac{\partial \varphi_r}{\partial x_\nu} - g^{\mu\nu}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_\alpha)}\partial^\nu \psi_\alpha + \partial^\nu \bar\psi_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar\psi_\alpha)} - g^{\mu\nu}\mathcal{L} =$$

$$= \frac{i}{2}\bar\psi \gamma^\mu \partial^\nu \psi - \frac{i}{2}\partial^\nu \bar\psi \gamma^\mu \psi - g^{\mu\nu}\mathcal{L}$$

If ψ and $\bar\psi$ satisfy the equations of motion $\mathcal{L} = \frac{1}{2}\bar\psi(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{2}(\partial_\mu\bar\psi\gamma^\mu + m\bar\psi)\psi = 0$

and

$$\Theta^{\mu\nu}(x) = \frac{i}{2}\left(\bar\psi(x)\gamma^\mu\partial^\nu\psi(x) - \partial^\nu\bar\psi(x)\gamma^\mu\psi(x)\right)$$

Then the four-vector associated to Dirac field is:

$$P^\nu = \int d^3x\, \Theta^{0\nu}$$

Noether currents associated to Lorentz transformations invariance

$$\mathcal{W}^{\mu(\rho\sigma)}(x) = \mathcal{M}^{\mu(\rho\sigma)}(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\alpha)}\left(I^{(\rho\sigma)}\right)^\alpha_\beta \psi_\beta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi_\alpha)}\left(I^{(\rho\sigma)}\right)^\alpha_\beta \bar\psi_\beta + \Theta^{\mu\sigma}x^\rho - \Theta^{\mu\rho}x^\sigma$$

and since $\left(I^{(\rho\sigma)}\right)^\alpha_\beta = -i\left(J^{\rho\sigma}\right)^\alpha_\beta$; $J^{\rho\sigma} = \frac{i}{4}[\gamma^\rho, \gamma^\sigma]$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi_\alpha)} = -\frac{i}{2}\gamma^\mu \cancel{\psi} \qquad , \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\alpha)} = \frac{i}{2}\bar\psi\gamma^\mu$$

results:

doua comutari

$$\mathcal{W}^{\mu(\rho\sigma)}(x) = \frac{i}{2}\bar\psi\gamma^\mu(-i)\frac{i}{4}[\gamma^\rho,\gamma^\sigma]\psi + \bar\psi(+i)\frac{i}{4}[\gamma^\rho,\gamma^\sigma]\left(-\frac{i}{2}\right)\gamma^\mu\psi + \Theta^{\mu\sigma}x^\rho - \Theta^{\mu\rho}x^\sigma .$$

$$= \frac{i}{4}\bar\psi\gamma^\mu[\gamma^\rho,\gamma^\sigma]\psi + \Theta^{\mu\sigma}x^\rho - \Theta^{\mu\rho}x^\sigma$$

Observation: $S^{i0} = \frac{i}{4}[\gamma^i,\gamma^0] = \frac{i}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix} = i\Sigma^{i0}$

$$S^{ij} = \frac{i}{4}[\gamma^i,\gamma^j] = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix}+\sigma_k & 0\\ 0 & +\sigma_k\end{pmatrix} = \frac{1}{2}\varepsilon^{ijk}\Sigma^k \equiv i\Sigma^{ij}$$

Then

$\mathcal{W}^{\mu(\rho\sigma)}$

and up to a divergence

$$\mathcal{M}^{\mu(\rho\sigma)} = i\,\bar\psi\,\gamma^\mu\Big(\big(x^\rho\partial^\sigma - x^\sigma\partial^\rho\big) + \Sigma^{\rho\sigma}\Big)\psi$$

$$\Sigma^{\rho\sigma} = \frac{1}{4}[\gamma^\rho,\gamma^\sigma] \quad , \quad \Sigma^{jk} = \frac{1}{2i}\,\varepsilon^{jkl}\,\Sigma^l$$

$$\Sigma^l = \begin{pmatrix}\sigma^l & 0\\ 0 & \sigma^l\end{pmatrix}$$

Now $\quad M^{\rho\sigma} = \int d^3x\, \mathcal{M}^{0\rho\sigma}$ and

$$\vec J = \frac{\vec e_i}{2}\varepsilon^{ijk} M^{jk} = \frac{\vec e_i}{2}\varepsilon^{ijk}\int d^3x\; i\,\psi^\dagger \underbrace{\gamma^0\gamma^0}_{I}\big(x^j\partial^k - x^k\partial^j + \Sigma^{jk}\big)\psi$$

$$= \vec e_i\, i\int d^3x\,\psi^\dagger\,(\vec x\times(-\vec\nabla))^i\,\psi + \frac{\vec e_i}{2}\int d^3x\,\psi^\dagger\, i\,\varepsilon^{ijk}\frac{1}{2i}\varepsilon^{jkl}\Sigma^l\psi$$

$$= \vec e_i \int d^3x\,\psi^\dagger\Big(\vec x\times\frac{1}{i}\vec\nabla\Big)^i\psi + \frac{\vec e_i}{2}\int d^3x\,\psi^\dagger\,\Sigma^i\,\psi$$

since $\quad \varepsilon^{ijk}\varepsilon^{ljk} = 2\delta^{il}$

Here the orbital angular momentum is

$$\vec L = \int d^3x\,\psi^\dagger\Big(\vec x\times\frac{1}{i}\vec\nabla\Big)\psi$$

and the spin angular momentum

$$\vec S = \frac{1}{2}\int d^3x\,\psi^\dagger\,\vec\Sigma\,\psi \qquad \text{with} \quad \vec\Sigma = \begin{pmatrix}\vec\sigma & 0\\ 0 & \vec\sigma\end{pmatrix}$$

Dirac Field: Momentum representation. Positive and negative frequency decomposition

Let us summarize the basic informations we need:

- the Dirac spinor $\psi_D(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$ obeys the Dirac equation

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi_D(x) = 0 \quad ; \text{ or in components } \quad (i\gamma^\mu\partial_\mu - m)_{\alpha\beta}\psi_\beta = 0$$

The equation for the adjoint $\bar{\psi}(x) = \psi^+(x)\gamma^0$ is

$$i\partial_\mu \bar{\psi}_D(x)\gamma^\mu + m\bar{\psi}_D(x) = 0$$

- observe that $(\gamma^\mu p_\mu)^2 = p_\nu p^\nu$

indeed: $2(\gamma^\mu p_\mu)^2 = \gamma^\mu p_\mu \gamma^\nu p_\nu + \gamma^\nu p_\nu \gamma^\mu p_\mu = p_\mu p_\nu \{\gamma^\mu, \gamma^\nu\} = 2 p_\mu p_\nu g^{\mu\nu} = 2 p_\mu p^\mu$ q.e.d.

(1) Introduce the Fourier decomposition of the spinor as:

$$\psi(x) = \frac{1}{(2\pi)^2}\int d^4k\, e^{ik_\mu x^\mu}\tilde{\psi}(k)$$

Since each component $\psi_\alpha(x)$ verify the Klein-Gordon equation the Fourier component $\tilde{\psi}(k)$ can be written as

$$\tilde{\psi}(k) = \sqrt{2\pi}\,\delta(k^2 - m^2)\psi(k)$$

(See the arguments from scalar field)

Then $\psi(x) = \frac{1}{(2\pi)^2}\sqrt{2\pi}\int d^4k\, e^{ik_\mu x^\mu}\delta(k^2 - m^2)\psi(k) =$

$$= \frac{1}{(2\pi)^{3/2}}\int d^4k\, e^{ikx}\delta(k^2-m^2)\left[\theta(k_0) + \theta(-k_0)\right]\psi(k) = \frac{1}{(2\pi)^{3/2}}\int dk_0 d^3k\, e^{ikx}\delta(k^2-m^2)\theta(k_0)\phi(k) +$$

$$+ \frac{1}{(2\pi)^{3/2}}\int dk_0 d^3k\, e^{ikx}\theta(-k_0)\phi(k) =$$

For the second term transform $k \to -k$

$$= \frac{1}{(2\pi)^{3/2}}\int dk_0\, d^3k\, e^{ikx}\delta(k^2-m^2)\theta(k_0)\phi(k) + \frac{1}{(2\pi)^{3/2}}\int dk_0\, d^3k\, e^{-ikx}\delta(k^2-m^2)\theta(k_0)\phi(-k) =$$

$$\equiv \psi^+(x) + \psi^-(x)$$

$\psi^+(x) \to$ the positive frequency part of the Dirac field $\psi(x)$

$\psi^-(x) \to$ the negative frequency part of the Dirac field $\psi(x)$

If we employ the decomposition: $\delta(xy)=\frac{1}{|x|}\delta(y)+\frac{1}{|y|}\delta(x)$ (6D)

$$\delta(k^2-m^2)=\delta(k_0^2-\vec{k}^2-m^2)=\delta\left((k^0-\sqrt{\vec{k}^2+m^2})(k^0+\sqrt{\vec{k}^2+m^2})\right)=\frac{1}{2\sqrt{\vec{k}^2+m^2}}\left(\delta(k^0-\sqrt{\vec{k}^2+m^2})+\delta(k^0+\sqrt{\vec{k}^2+m^2})\right)$$

then with $\omega_p=\sqrt{\vec{k}^2+m^2}>0$ and $\psi^+(\vec{k})=\frac{\psi(k)}{\sqrt{2\omega_p}}=\frac{\psi(\omega_p,\vec{k})}{\sqrt{2\omega_p}}$

$$\psi^+(x)=\frac{1}{(2\pi)^{3/2}}\int dk_0 d^3k\, e^{ikx}\delta(k^2-m^2)\theta(k_0)\psi(k)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{\sqrt{2\omega_p}}e^{i(\omega_p x^0-\vec{k}\cdot\vec{x})}\psi^+(\vec{k},\omega_p)$$

and with $\psi^-(\vec{k})=\frac{\psi(-k)}{\sqrt{2\omega_p}}=\frac{\psi(-\omega_p,-\vec{k})}{\sqrt{2\omega_p}}$

$$\psi^-(x)=\frac{1}{(2\pi)^{3/2}}\int dk_0 d^3k\, e^{-ikx}\delta(k^2-m^2)\theta(k_0)\psi(-k)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{\sqrt{2\omega_p}}e^{-i(\omega_p x^0-\vec{k}\cdot\vec{x})}\psi^-(\vec{k},\omega_p)$$

and so (since $\omega_p=\omega_p(\vec{k})$)

$$\psi(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{\sqrt{(2\omega_p)}}\left(e^{-i(\omega_p x^0-\vec{k}\cdot\vec{x})}\psi^-(\vec{k})+e^{i(\omega_p x^0-\vec{k}\cdot\vec{x})}\psi^+(\vec{k})\right)$$

(2) Let us now observe that since $(i\gamma^\mu\partial_\mu-m)\psi(x)=0\Rightarrow$

$$\frac{1}{(2\pi)^2}\int d^4k\,(i\gamma^\mu ik_\mu-m)\tilde{\psi}(k)e^{ik_\mu x^\mu}=0\Rightarrow\qquad(\gamma^\mu k_\mu+m)\tilde{\psi}(k)=0$$

$$\sqrt{2\pi}\,\delta(k^2-m^2)(\gamma^\mu k_\mu+m)\psi(k)=0$$

or equivalently
$$(\gamma^\mu k_\mu+m)\psi(k)\Big|_{k^2=m^2}=0$$

Now for $\psi^+(\vec{k})=\frac{\psi(k)}{\sqrt{2\omega_p}}$ results from the above result that $\psi^+(\vec{k})$ obey the matrix equations

$$\boxed{(\gamma^\mu k_\mu+m)\psi^+(\vec{k})=0}$$

while $\psi^-(\vec{k})=\frac{\psi(-k)}{\sqrt{2\omega_p}}=\frac{\psi(-\omega_p,-\vec{k})}{2\omega_p}$ satisfy the matrix equations

$$\boxed{(\gamma^\mu k_\mu-m)\psi^-(\vec{k})=0}$$

The equations satisfied by $\psi^+(\vec{k})$ and $\psi^-(\vec{k})$ can be written in the form

$$(\pm\gamma^0p^0-\vec{\gamma}\cdot\vec{p}+m)\psi^\pm($$

The equation $\boxed{(\gamma^\mu k_\mu - m)\psi^-(\vec{k}) = 0}$ → for negative frequency compo (positive energy solutions)

Using chiral representation we write (with $u(p) = \psi^-(\vec{k})$) in the rest frame where $p^\mu = (m,0,0,0)$: $(\gamma^0 - 1)u(p) = 0 \Rightarrow \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} u_L \\ u_R \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_L \\ u_R \end{pmatrix} = 0$

$\begin{cases} -u_L + u_R = 0 \\ u_L - u_R = 0 \end{cases} \Rightarrow u_L = u_R$ | If $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & -2I \end{pmatrix}\psi = 0$ i.e. 2 of the 4 comp in ψ are zero

Comment: while KG equation imposes only the mass shell condition $k^2 = m^2$, the Dirac equation, being first-order in the derivatives, has also the effect of reducing by a factor of two the number of independent degrees of freedom

We have two linear-independent solutions $u_s(k)$ with $s = 1,2$ and the most general solution $\psi^-(\vec{k})$ is a superposition of them

$$\psi^-(\vec{k}) = \sum_s u_s(k)\, a_s^-(\vec{k})$$

The equation $\boxed{(\gamma^\mu k_\mu + m)\psi^+(\vec{k}) = 0}$ for positive frequency comp (negative energy solutions)

Using chiral representation (with $v(p) = \psi^+(\vec{k})$) in the rest frame

$$(\gamma^0 + 1)v(p) = 0 \Rightarrow \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} v_L \\ v_R \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_L \\ v_R \end{pmatrix} = 0$$

$\begin{cases} v_L + v_R = 0 \\ v_L + v_R = 0 \end{cases}$ $v_L = -v_R$

Again we have two independent solutions $v_s(k)$ with $s = 1,2$

$$\psi^+(\vec{k}) = \sum_s v_s(k)\, b_s^+(\vec{k})$$

Therefore, in terms of these four-components spinors $u_s(\vec{k})$, $v_s(\vec{k})$ the general expansion of the Dirac field is

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_p}} \sum_{s=1,2} \left(a_s^-(\vec{k})\, u_s(\vec{k}) e^{-ikx} + b_s^+(\vec{k})\, v_s(\vec{k}) e^{ikx} \right)$$

Dynamical invariants for Dirac fields

The energy-momentum four-vector

$$P^{\nu} = \int d^3x \, \Theta^{0\nu}$$

where (from $\Theta^{\mu\nu}(x) = \frac{i}{2}\left(\bar{\psi}(x)\gamma^{\mu}\partial^{\nu}\psi(x) - \partial^{\nu}\bar{\psi}(x)\gamma^{\mu}\psi(x)\right)$)

$$\Theta^{0\nu} = \frac{i}{2}\left(\bar{\psi}(x)\gamma^{0}\partial^{\nu}\psi(x) - \partial^{\nu}\bar{\psi}(x)\gamma^{0}\psi(x)\right)$$

Now with frequency decomposition

$$\psi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\omega_p}} \sum_{s=1,2}\left(a_s^-(\vec{k})u_s(\vec{k})e^{-ikx} + b_s^+(\vec{k})v_s(\vec{k})e^{ikx}\right)$$

$$\bar{\psi}(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\omega_p}} \sum_{s=1,2}\left((b_s^+(\vec{k}))^* \bar{v}_s(\vec{k})e^{-ikx} + (a_s^-(\vec{k}))^* \bar{u}_s(\vec{k})e^{ikx}\right)$$

Now

$$\partial^{\nu}\psi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\omega_p}}\left((-ik^{\nu})\sum_{s=1,2} a_s^-(\vec{k})u_s(\vec{k})e^{-ik^{\mu}x_{\mu}} + (ik^{\nu})\sum_{s=1,2} b_s^+(\vec{k})v_s(\vec{k})e^{ik^{\mu}x_{\mu}}\right)$$

$$\partial^{\nu}\psi^{+}(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\omega_p}}\left((-ik^{\nu})\sum_{s=1,2} (b_s^+(\vec{k}))^* v_s^+(\vec{k})e^{-ik^{\mu}x_{\mu}} + (ik^{\nu})\sum_{s=1,2} (a_s^-(\vec{k}))^* u_s^+(\vec{k})e^{ik^{\mu}x_{\mu}}\right)$$

Then, since $\gamma^0\gamma^0 = I$:

$$\int dy\, e^{i(z-z')y} = 2\pi\,\delta(z-z') \Rightarrow \int d^3x\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = (2\pi)^3\delta(\vec{k}-\vec{k}')$$

$$u_s^+(\vec{k})\, v_{s'}(-\vec{k}) = 0$$

$$v_s^+(\vec{k})\, u_{s'}(-\vec{k}) = 0$$

$$u_s^+(\vec{k})\, u_{s'}(\vec{k}) = 2\omega_p\,\delta_{ss'}$$

$$v_s^+(\vec{k})\, v_s(\vec{k}) = 2\omega_p\,\delta_{ss'}$$

$$\bar{\psi}\gamma^0\partial^{\nu}\psi = \psi^+\gamma^0\gamma^0\partial^{\nu}\psi = \psi^+\partial^{\nu}\psi = \frac{1}{(2\pi)^3}\int\frac{d^3k\,d^3k'}{2\sqrt{\omega_p\omega_{p'}}}\sum_{s,s'}\left((b_{s'}^+(\vec{k}'))^* v_{s'}^+(\vec{k}')e^{-ik'x} + (a_{s'}^-(\vec{k}'))^* u_{s'}^+(\vec{k}')e^{ik'x}\right)\left((-ik^{\nu})a_s^-(\vec{k})u_s(\vec{k})e^{-ikx} + (ik^{\nu})b_s^+(\vec{k})v_s(\vec{k})e^{ikx}\right)$$

Then

$$(\partial^{\nu}\psi^+)\psi = \frac{1}{(2\pi)^3}\int\frac{d^3k\,d^3k'}{2\sqrt{\omega_p\omega_{p'}}}\sum_{s,s'}\left((-ik'^{\nu})(b_{s'}^+(\vec{k}'))^* v_{s'}^+(\vec{k}')e^{-ik'x} + (ik'^{\nu})(a_{s'}^-(\vec{k}'))^* u_{s'}^+(\vec{k}')e^{ik'x}\right)\left(a_s^-(\vec{k})u_s(\vec{k})e^{-ikx} + b_s^+(\vec{k})v_s(\vec{k})e^{ikx}\right)$$

Then

$$P^{\nu} = \int d^3x\,\Theta^{0\nu} = \frac{i}{2}\left(\int d^3x\,\psi^+\partial^{\nu}\psi - \int d^3x\,(\partial^{\nu}\psi^+)\psi\right)$$

Warm

$$\frac{i}{2}\int d^3x\,(\partial^\nu\psi^+)\psi = \frac{i}{2}\frac{1}{(2\pi)^3}\int\frac{d^3k\,d^3k'}{2\sqrt{\omega_p\omega_{p'}}}\sum_{s,s'}\Big((-ik'^\nu)\,(b^+_{s'}(\vec{k}'))^*a^-_s(\vec{k})\,v^+_{s'}(\vec{k}')u_s(\vec{k})\int d^3x\,e^{i(\vec{k}'+\vec{k})\cdot\vec{x}}e^{-i(\omega_p+\omega_{p'})x^0}$$

$$(-ik'^\nu)\,(b^+_{s'}(\vec{k}'))^*b^+_s(\vec{k})\,v^+_{s'}(\vec{k}')v_s(\vec{k})\int d^3x\,e^{i(\vec{k}'-\vec{k})\cdot\vec{x}}e^{-i(\omega_{p'}-\omega_p)x^0}$$

$$+(ik'^\nu)\,(a^-_{s'}(\vec{k}'))^*a^-_s(\vec{k})\,u^+_{s'}(\vec{k}')u_s(\vec{k})\int d^3x\,e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}}e^{i(\omega_p'-\omega_p)x^0}$$

$$+(ik'^\nu)\,(a^-_{s'}(\vec{k}'))^*b^+_s(\vec{k})\,u^+_{s'}(\vec{k}')v_s(\vec{k})\int d^3x\,e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}}e^{i(\omega_p+\omega_{p'})x^0}\Big)$$

$$=\frac{i}{2}\frac{(2\pi)^3}{(2\pi)^3}\int\frac{d^3k\,d^3k'}{2\sqrt{\omega_p\omega_{p'}}}\sum_{s,s'}(-ik'^\nu)\delta(\vec{k}'+\vec{k})\,(b^+_{s'}(\vec{k}'))^*a^-_s(\vec{k})\,v^+_{s'}(\vec{k}')u_s(\vec{k})\,e^{-i(\omega_p+\omega_{p'})x^0}$$

$$(-ik'^\nu)\delta(\vec{k}'-\vec{k})\,(b^+_{s'}(\vec{k}'))^*b^+_s(\vec{k})\,v^+_{s'}(\vec{k}')v_s(\vec{k})\,e^{-i(\omega_p-\omega_{p'})x^0}$$

$$(+ik'^\nu)\,\delta(\vec{k}'-\vec{k})\,(a^-_{s'}(\vec{k}'))^*a^-_s(\vec{k})\,u^+_{s'}(\vec{k}')u_s(\vec{k})\,e^{i(\omega_p'-\omega_p)x^0}$$

$$+(ik'^\nu)\,\delta(\vec{k}'+\vec{k})\,(a^-_{s'}(\vec{k}'))^*b^+_s(\vec{k})\,u^+_{s'}(\vec{k}')v_s(\vec{k})\,e^{i(\omega_p+\omega_{p'})x^0}=$$

$$=-\frac{i^2}{2}\int\frac{d^3k}{2\omega_p}\sum_{s,s'}\Big((-k^\nu)\,(b^+_{s'}(-\vec{k}))^*a^-_s(\vec{k})\,v^+_{s'}(-\vec{k})u_s(\vec{k})\,e^{-2i\omega_p x^0}$$

$$(k^\nu)\,(b^+_{s'}(\vec{k}))^*b^+_s(\vec{k})\,v^+_{s'}(\vec{k})v_s(\vec{k})$$

$$(-k^\nu)\,(a^-_{s'}(\vec{k}))^*a^-_s(\vec{k})\,u^+_{s'}(\vec{k})u_s(\vec{k})$$

$$(k^\nu)\,(a^-_{s'}(-\vec{k}))^*b^+_s(\vec{k})\,u^+_{s'}(-\vec{k})v_s(\vec{k})\,e^{2i\omega_p x^0}\Big)$$

$$=\frac{1}{2}\int\frac{d^3k}{\cancel{2\omega_p}}\sum_s(-k^\nu)\Big((a^-_s(\vec{k}))^*a^-_s(\vec{k})-(b^+_s(\vec{k}))^*b^+_s(\vec{k})\Big)\cancel{2\omega_p}$$

$$=-\frac{1}{2}\int d^3k\sum_s k^\nu\Big((a^-_s(\vec{k}))^*a^-_s(\vec{k})-(b^+_s(\vec{k}))^*(b^+_s(\vec{k}))\Big)$$

SCALAR FIELD

- For a real scalar field $\phi(x)$ the commutation relations in momentum ~~space~~ representation are:

$$[\phi^-(\vec{k}), \phi^+(\vec{q})] = \delta(\vec{k}-\vec{q})$$

$$[\phi^+(\vec{k}), \phi^+(\vec{q})] = 0 \; ; \; [\phi^-(\vec{k}), \phi^-(\vec{q})] = 0$$

while for a complex scalar field, described by $\phi(x)$ and $\phi^*(x)$

$$[\phi^{*-}(\vec{k}), \phi^+(\vec{q})] = [\phi^-(\vec{k}), \phi^{*+}(\vec{q})] = \delta(\vec{k}-\vec{q})$$

and all the other commutators are zero

In configuration space (for real scalar field)

$$[\phi^-(x), \phi^+(y)] = \frac{1}{(2\pi)^3}\int \frac{d^3k\, d^3q}{\sqrt{2\omega_p}\sqrt{2\omega_{p'}}} e^{i(ky-qx)} \delta(\vec{k}-\vec{q}) =$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega_p} e^{ik(y-x)} \equiv \frac{1}{i} D^-(\cancel{x}-\cancel{y})$$

$$[\phi^+(x), \phi^-(y)] = \frac{1}{(2\pi)^3}\int \frac{d^3k\, d^3q}{\sqrt{2\omega_p}\sqrt{2\omega_p}} e^{i(kx-qy)} (-)\delta(\vec{k}-\vec{q}) =$$

$$= \frac{-1}{(2\pi)^3}\int \frac{d^3k}{2\omega_p} e^{ik(x-y)} \equiv -i D^+(x-y)$$

where was used the definition

$$\frac{1}{(2\pi)^3}\int \frac{d^3\vec{k}}{2\omega_p} e^{ik(x-y)} = \frac{1}{i} D^-(y-x) = i D^+(x-y)$$

and the so-called Pauli-Jordan commutation function

$$D(x) = D^+(x) + D^-(x) = i \frac{1}{(2\pi)^3}\int e^{-ik_\mu x^\mu} \epsilon(k_0)\delta(k^2-m^2)\, d^4k \, .$$

Since the function $D(x)$ vanishes outside the light cone and

$$[\varphi(x), \varphi(y)] = -i D(x-y)$$

is concluded that $[\varphi(x), \varphi(y)] = 0$ if $(x-y)^2 < 0$ i.e. we have the causal independence of events separated by spacelike intervals

CANONICAL QUANTIZATION OF MECHANICAL UNCONSTRAINED SYSTEMS

SUMMARY

① Lagrangian Formalism

$$q^i ; \dot{q}^i \to L(q^i, \dot{q}^i) \to \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0$$

② The Hamiltonian

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \to \text{canonical momentum} \qquad H = p_i \dot{q}^i - L \Rightarrow \begin{cases} \dot{q}^i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q^i} \end{cases}$$

③ Poisson Brackets $A = A(p_i, q^i)$; $B = B(p_i, q^i)$

$$\{A,B\} = \left(\frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i}\right) \Rightarrow \begin{cases} \{AB,C\} = A\{B,C\} + \{A,C\}B \\ \{A,B\} = -\{B,A\} \\ \{q^i, p_j\} = \delta^i_j \,;\; \{q^i,q^j\} = \{p_i,p_j\} = 0 \end{cases}$$

④ Quantization $\{A,B\}_{cn} = \{A,B\}_{qn}$ $\qquad \{A,B\}_{qn} = -\frac{i}{\hbar}[A,B] \Rightarrow [A,B] = i\hbar\{A,B\}_{qn}$

In Heisenberg formulation: $i\hbar\dot{A} = [A,H]$

IN FIELD THEORY

$\phi(x) = \phi(\vec{x},t) \to$ describing one degree of freedom at each point of space

$\vec{x} \to$ continuous extension of the discrete index i ; summation $\to$ extends to integrals

$$\begin{cases} i \to \vec{x} \\ j \to \vec{y} \\ \sum_i \to \int d^3x \\ \delta_{ij} \to \delta^3(\vec{x}-\vec{y}) \end{cases} \quad ; \quad L(t) = \int d^3x\, \mathcal{L}(t,\vec{x}) \leftarrow \text{Lagrangian density}$$

Euler-Lagrange equations: $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0$

The variable canonically conjugated to $u_i(x)$ is $\pi^i(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 u_i)}$

$\mathcal{H}(x) = \pi^i(x)\dot{u}_i(x) - \mathcal{L} = \mathcal{H}(u_i(x), \pi^i(x)) \to$ Hamiltonian density

$H = \int d^3x\, \mathcal{H}(x) \to$ the total Hamiltonian

Hamilton equations: $\dot{u}_i = \frac{\partial \mathcal{H}}{\partial \pi^i}$; $\dot{\pi}^i = -\frac{\partial \mathcal{H}}{\partial u_i}$

Poisson Brackets:

$$\{A(x), B(y)\}_{x_0=y_0} = \left[\frac{\partial A(x)}{\partial u_i(x)}\frac{\partial B(y)}{\partial \pi^i(y)} - \frac{\partial A(x)}{\partial \pi^i(x)}\frac{\partial B(y)}{\partial u_i(y)}\right]\delta^3(\vec{x}-\vec{y})$$

equal-time brackets

Now the Poisson brackets between canonical equal-time variables read:

$$\{u_i(x), \pi^j(y)\}_{x_0=y_0} = \delta_i^j\,\delta^3(\vec{x}-\vec{y})$$

$$\{u_i(x), u_j(y)\}_{x_0=y_0} = \{\pi^i(x), \pi^j(y)\}_{x_0=y_0} = 0$$

QUANTIZATION

Free scalar field: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$; $\pi = \dfrac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi$ Functional derivative

$$\{\phi(x), \phi(y)\}_{x_0=y_0} = \{\pi(x), \pi(y)\}_{x_0=y_0} = 0 \; ; \; \{\phi(x), \pi(y)\}_{x_0=y_0} = \delta^{(3)}(\vec{x}-\vec{y})$$

Comment: the scalar field is the unique field theory which can be quantized in the simple way described in the previous section; other systems, including the free Dirac Lagrangian, imply algebraic relations between fields and their conjugate momenta $\longrightarrow$ CONSTRAINTS

The relativistic scheme of quantized fields

- as a result of quantization the FIELDS acquire an operator meaning and are expressed linearly in terms of the particle creation and annihilation operators between which required commutation relations are set-up $\phi(x) \longrightarrow \hat{\phi}(x)$

$\boxed{\text{The fundamental postulate for fields quantization}}$: the Hermitian operators for the energy-mom. four-vector P^{ν}, the angular momentum tensor $M^{\mu\nu}$, the charge ϕ which are the generators of infinitesimal transformations of the state vectors associated to the Poincaré-Lorentz and gauge transformations are expressed in terms of the operator fields functions by the same relations as in classical field theory.

$\boxed{\text{Obs}}$ the operator factors should be arranged in appropriate order

Let be $|\Phi\rangle$ the state vector of the field system (for all real physical states the norms of the ~~physical~~ vectors is finite (one))

$\boxed{\text{For a classical transformation}}$: $x \to x' = f(\omega, x) = \Lambda x$

$$\phi'(x') = U_c(\omega)\, \phi(x)$$

will correspond an $\boxed{\text{unitary transformation}}$ $U(\omega)$ (norm conservation & superposition principle)

$$U_c(\omega) \to U(\omega) \qquad \text{with } U^{+} = U^{-1}$$

and

$$|\Phi'\rangle = U(\omega)\,|\Phi\rangle \quad \to \text{state transformation in quantum case}$$

$\boxed{\text{Is POSTULATED}}$: that the matrix elements ~~between the primed and unprimed~~ of the field operators transforms analogously as the corresponding classical field

Denote $U_c(\omega) \equiv M(\omega)$. Then

$$\langle \beta' | \hat{\phi}(x') | \alpha' \rangle = M(\omega) \langle \beta | \hat{\phi}(x) | \alpha \rangle$$

Comment: • as in quantum mechanics if the state is transformed, we consider that the operators are not changing

$$\begin{cases} |\alpha'\rangle = U(\omega)|\alpha\rangle \\ |\beta'\rangle = U(\omega)|\beta\rangle \\ x' = \Lambda(\omega)x \end{cases}$$

• specifying the set of field operators $\hat{\phi}(x)$ at all points x of the Minkowski space ~~we~~ describe completely the degrees of freedom of the quantum field

As in Dirac, a change of state is equivalent with the transformation of the field operators:

$$U^{-1}(\omega)\hat{\phi}(x')U(\omega) = M(\omega)\phi(x)$$

or

$$\boxed{U^{-1}(\omega)\hat{\phi}(x)U(\omega) = M(\omega)\phi(\Lambda^{-1}x)}$$

For continuous infinitesimal transformations (Poincaré Lorentz transf)

$$\delta x^\mu = \varepsilon^\mu + \omega^\mu{}_\nu x^\nu$$

$$G(\varepsilon^\mu, \omega^{\mu\nu}) = -\varepsilon_\mu P^\mu + \frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}$$

and the ansatz for the infinitesimal unitary transformation is

$$U(\varepsilon,\omega) = I + i\varepsilon_\mu P^\mu - \frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu}$$

The quantities P^μ and $J^{\mu\nu}$ are identified as momentum vector operator and the angular momentum tensor operator respectively.

Now from $U^{-1}\phi(x)U = M(\omega)\phi(\Lambda^{-1}x) \Rightarrow$

$$(1+iG)\phi(1-iG) = M(\omega)\phi(x-\delta x) \Rightarrow$$

$$\phi + iG\phi - i\phi G + \underbrace{G\phi G}_{O(\omega^2)} = \underbrace{(M(\omega)-I)}_{\sim\omega}\Big(\hat{\phi}(x) - \underbrace{\partial_\mu\hat{\phi}(x)\,\delta x^\mu}_{\sim\omega}\Big) + I\big(\hat{\phi}(x) - \partial_\mu\hat{\phi}(x)\delta x^\mu\big)$$

To first order

$$\phi(x) + i[G(\omega),\hat{\phi}(x)] = \phi(x) - \delta x^\mu\partial_\mu\hat{\phi}(x) + (M(\omega)-I)\hat{\phi}(x) \quad \text{or}$$

$$i[G(\omega),\hat{\phi}(x)] = -\delta x^\mu\partial_\mu\hat{\phi}(x) + (M(\omega)-I)\hat{\phi}(x)$$

For pure translations

$$\omega^{\mu\nu} = 0 \qquad G_\mu = -\epsilon_\mu P^\mu \qquad \delta x_\mu = \epsilon_\mu$$

$$i\left[-\epsilon_\mu P^\mu, \hat{\phi}(x)\right] = -\epsilon_\mu \partial^\mu \hat{\phi}(x) \quad \text{or} \quad -\left[P^\mu, \hat{\phi}(x)\right] = i\partial^\mu \phi(x)$$

$$\boxed{i\partial^\mu \phi = [\phi, P^\mu]}$$

(Directly we can write

$$(1 - i\epsilon_\mu P^\mu)\phi(x)(1 + i\epsilon_\mu P^\mu) = \phi(x - \epsilon^\mu)$$

$$\phi(x) - i\epsilon_\mu [P^\mu, \phi] = \phi(x) - \epsilon^\mu \partial_\mu \phi \Rightarrow i[P^\mu, \phi] = \partial^\mu \phi$$)

For Lorentz transformations

the generator of Lorentz transf (four-dim relation)

$$U(\omega) = I - \frac{i}{2}\omega_{\mu\nu} J^{\mu\nu} \quad ; \quad M(\omega) = I - \frac{i}{2}\omega_{\mu\nu} I^{\mu\nu}$$

$$\left(1 + \frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}\right)\phi(x)\left(1 - \frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}\right) = (M(\omega) - I)(\phi(x) - \partial_\mu \phi\, \omega^\mu{}_\nu x^\nu) + I\underbrace{(\phi(x) - \partial_\mu \phi\, \omega^\mu{}_\nu x^\nu)}_{\phi(x - \delta x)}$$

Then (in first order)

$$\phi(x) - \frac{i}{2}\omega_{\mu\nu}[\hat{\phi}(x), J^{\mu\nu}] = \phi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi - \frac{i}{2}\omega_{\mu\nu} I^{\mu\nu} \hat{\phi}(x)$$

$$-\frac{i}{2}\omega_{\mu\nu}[\hat{\phi}(x), J^{\mu\nu}] = -\frac{i}{2}\omega_{\mu\nu} I^{\mu\nu}\phi(x) - x^\nu \omega_{\rho\nu} g^{\rho\mu}\partial_\mu \phi$$

$$-\frac{i}{2}\omega_{\mu\nu}[\hat{\phi}(x), J^{\mu\nu}] = -\frac{i}{2}\omega_{\mu\nu} I^{\mu\nu}\phi(x) - \omega_{\mu\nu} x^\nu \partial^\mu \phi$$

$$[\hat{\phi}(x), J^{\mu\nu}] = I^{\mu\nu}\phi(x) - i(\omega_{\mu\nu} x^\nu \partial^\mu - \omega_{\nu\mu} x^\nu \partial^\mu)\phi$$

$$\boxed{[\hat{\phi}(x), J^{\mu\nu}] = I^{\mu\nu}\phi(x) + i(x^\mu \partial^\nu - x^\nu \partial^\mu)\phi}$$

Similarly, for Gauge Transformation (phase transformations)

$$\phi' = e^{-iq\alpha}\phi \quad \Rightarrow \quad |\Phi'\rangle = U(\alpha)|\Phi\rangle$$

$$\phi'^* = e^{+iq\alpha}\phi^* \qquad U(\alpha) = e^{-i\alpha Q}$$

... $U^{-1}\phi U = e^{-i\alpha q}\phi$ ... $\boxed{[\phi, Q] = \phi}$

The physical meaning of frequency components

- From the frequency decomposition

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_p}} \left(\phi^+(\vec{k}) e^{ik_\mu x^\mu} + \phi^-(\vec{k}) e^{-ik_\mu x^\mu} \right)$$

$\phi^+(\vec{k})$; $\phi^-(\vec{k})$ – operat.

since $i\partial_\mu \phi = [\phi(x), P_\mu]$ results

$$\partial_\mu \phi^+(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_p}} \phi^+(\vec{k})\, ik_\mu\, e^{ik_\mu x^\mu}$$

$$\partial_\mu \phi^-(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_p}} \phi^-(\vec{k})\, (-ik_\mu)\, e^{-ik_\mu x^\mu}$$

$$[\phi^\pm(x), P_\mu] = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_p}} [\phi^\pm(\vec{k}), P_\mu]\, e^{\pm ik_\mu x^\mu}$$

and we can conclude that

$$i(\pm i)k_\mu\, \phi^\pm(\vec{k}) = [\phi^\pm(\vec{k}), P_\mu] \quad \text{or}$$

$$\begin{cases} k_\mu\, \phi^-(\vec{k}) = [\phi^-(\vec{k}), P_\mu] \\ k_\mu\, \phi^+(\vec{k}) = -[\phi^+(\vec{k}), P_\mu] \end{cases}$$

- Let be a state $|\alpha_p\rangle$ with the property $\boxed{P_\nu |\alpha_p\rangle = p_\nu |\alpha_p\rangle}$ i.e. is an eigenstate of the energy-momentum four vector (remember that $[P_\mu, P_\nu]=0$). With the help of previous relations, when applied on $|\alpha_p\rangle$ results

$$k_\mu \phi^-(\vec{k})|\alpha_p\rangle = \phi^-(\vec{k}) P_\mu |\alpha_p\rangle - P_\mu \phi^-(\vec{k})|\alpha_p\rangle \quad \text{or}$$

$$P_\mu \left(\phi^-(\vec{k})|\alpha_p\rangle \right) = (p_\mu - k_\mu)\left(\phi^-(\vec{k})|\alpha_p\rangle \right)$$

i.e. the state $\phi^-(\vec{k})|\alpha_p\rangle$ is still an eigenstate of P_μ but with fourmomentum $p_\mu - k_\mu$ or $\boxed{\phi^-(\vec{k})|\alpha_p\rangle = |\alpha_{p-k}\rangle}$

Conclusion:

$\phi^-(\vec{k})$ → describes the annihilation of a particle of mass m and four-momentum k

$P_\mu P^\mu$ – mass operator

Analogously from

$$k_\mu \phi^+(\vec{k})|\alpha_p\rangle = -\phi^+(\vec{k}) P_\mu |\alpha_p\rangle + P_\mu \phi^+(\vec{k})|\alpha_p\rangle \quad \text{or}$$

$$P_\mu \left(\phi^+(\vec{k})|\alpha_p\rangle\right) = (k_\mu + p_\mu)\left(\phi^+(\vec{k})|\alpha_p\rangle\right) \quad \text{i.e.}$$

$\phi^+(\vec{k})$ describes the creation of a particle of mass m and four-momentum k

- Concerning the phase transformations let be the state $|\alpha_e\rangle$ such that $Q|\alpha_e\rangle = e|\alpha_e\rangle$

Then from $\phi = [\phi, Q]$ results that

$$\phi|\alpha_e\rangle = \phi Q|\alpha_e\rangle - Q\phi|\alpha_e\rangle \quad \text{or} \quad Q(\phi|\alpha_e\rangle) = (e-1)(\phi|\alpha_e\rangle)$$

i.e.

$\phi|\alpha_e\rangle$ is a state with the charge reduced by one (unit)

$\phi(x)$ is lowering the charge of the system by one unit

Analogously, from $\phi^+ = -[\phi^+, Q]$ we conclude that

$$\phi^+|\alpha_e\rangle = -\phi^+ Q|\alpha_e\rangle + Q\phi^+|\alpha_e\rangle \Rightarrow Q(\phi^+|\alpha_e\rangle) = (e+1)(\phi^+|\alpha_e\rangle)$$

i.e. $\phi^+(x)$ is raising the charge eigenvalue by one (unit)

- Vacuum definition.

Let be the state $|\text{vacuum}\rangle$, the ground state of the field system. Since there are no particles in the vacuum, the momentum of vacuum is zero, while the energy should be minimal (may also be set equal to zero)

Then $\phi^-(\vec{k})|\text{vacuum}\rangle = 0$

and also

$$\phi^-(x)|\text{vacuum}\rangle = 0$$

Commutation relations

Comment 1. The classical Poisson brackets for the field functions $\{u(x), u(y)\}$ then turn out to be certain functions of x and y, more precisely of the difference $x-y$, independent of u.

Starting with the correspondence principle, it is customary to assume in quantum theory of the free field that the commutation rule for the operator fields has the form:

(10B) $$\{u_a(x), u_b(y)\}_- \equiv [u_a(x), u_b(y)] = u_a(x)u_b(y) - u_b(y)u_a(x) = \Delta_{ab}(x-y)$$

However this commutation relation, turns out to be too stringent and does not embrance a number of physically important cases

(10F) $$\{u_a(x), u_b(y)\}_+ \equiv u_a(x)u_b(y) + u_b(y)u_a(x) = \Delta_{ab}(x-y)$$

i.e. now the anticommutator of two operators is a c-number

The (anti)commutation relation (10F) leads to the commutators of dynamical variables which are quadratic forms in the field operators, being expressed in terms of the commutation functions Δ → it is in this sense that should be interpreted the correspondence principle in this second case

$\boxed{\text{Comment very IMPORTANT}}$

The exact form of the commutation functions Δ for any arbitrary field is determined by the equations:

$$i\frac{\partial \phi_a}{\partial x^\mu} = [\phi_a(x), P_\mu]$$

$$[\phi_a(x), Q] = \phi_a(x); \quad -[\phi_a^\dagger(x), Q] = \phi_a^\dagger(x)$$

and by the structure of the energy operator of the given field.

8Q

Observation the commutation function of free fields depends only on the difference $x-y$ i.e.

$$\{\phi_a(x), \phi_b(y)\} = \Delta_{ab}(x-y) \qquad (*)$$

Reflects the translational invariance of commutation rel

Proof: Since the formulas for the Fourier transformation are linear, the commutators or anticommutators of the frequency components in the momentum representation $\phi^{\pm}(\vec{k})$ must also be c-numbers

Step 1. We shall first show that operators of the same frequency must strictly commute or anticommute

$$\{\phi_a^{\pm}(\vec{k}), \phi_b^{\pm}(\vec{k})\} = 0$$

$\{\ \}$ → denote either a commutator or an anticommutator

Proof: Let be $|\alpha_p\rangle$ such that $P_\nu|\alpha_p\rangle = p|\alpha_p\rangle$. Then let be the states

$$|\alpha_1\rangle = \phi_a^+(\vec{k})\,\phi_b^+(\vec{q})|\alpha_p\rangle$$

$$|\alpha_2\rangle = \phi_b^+(\vec{q})\,\phi_a^+(\vec{k})|\alpha_p\rangle$$

which in agreement with the properties proved previously should satisfy the equations

$$P_0|\alpha_1\rangle = (p_0 + k_0 + q_0)|\alpha_1\rangle$$

$$P_0|\alpha_2\rangle = (p_0 + k_0 + q_0)|\alpha_2\rangle$$

Then

$$P_0(|\alpha_1\rangle \pm |\alpha_2\rangle) = P_0\{\phi_a^+(\vec{k}), \phi_b^+(\vec{q})\}|\alpha_p\rangle = (p_0+k_0+q_0)\{\phi_a^+, \phi_b^+\}|\alpha_p\rangle$$

If we now assume that $\{\phi_a^+, \phi_b^+\}$ is a c-number differing from zero, then canceling it out, we obtain $P_0|\alpha_p\rangle = (p_0+k_0+q_0)|\alpha_p\rangle$ i.e. a contradiction with $P_0|\alpha_p\rangle = p_0|\alpha_p\rangle$. So we have

$$\{\phi_a^+, \phi_b^+\} = 0$$

and by hermitian conjugation analogously

$$\{\phi_a^-, \phi_b^-\} = 0$$

Step 2 Analogously it may be proven that $\{\phi_a^{\pm}(\vec{k}), \phi_b^{\mp}(\vec{q})\} = 0$ when $\vec{k} \neq \vec{q}$

Physics → the <u>acts of creation of particles of any arbitrary fields do not interfere with each other and do not affect other acts of particle annihilation or the creation and annihilation of particles with different momenta</u>

Now assume that in the general case $\{\phi_a^{\pm}(\vec{k}), \phi_b^{\mp}(\vec{q})\} = \delta(\vec{k}-\vec{q})$

Then in the configuration space is obtained $(*)$

CONCRETE FORM

As we mention to determine the commutation function it is necessary to refer to the explicit form of the operator of the energy-momentum four vector and use the equations $i\partial_\mu \phi_a = [\phi_a, P_\mu]$ and $\begin{cases}\dot\phi_a = [\phi_a, Q] \\ \dot\phi_a^+ = -[\phi_a^+, Q]\end{cases}$

respectively.

Now from $i\dfrac{\partial\phi_a(x)}{\partial x^\mu} = [\phi_a(x), P_\mu]$ from where result $k_\mu \phi^\mp(\vec k) = \mp[\phi^\mp(\vec k), P_\mu]$

with the energy momentum operator written in the form stands for fields of integral spin

$$P_\nu = \int d^3k\; k_\nu \sum_s \left(a_s^+(\vec k)\, a_s^-(\vec k) \pm b_s(\vec k) b_s^+(\vec k)\right)$$

stands for fields of half-integral spins

Comment: we have expressed $\phi^\mp(\vec k)$ and $\phi^{+\pm}(\vec k)$ in terms of the independent amplitudes

$$\phi_a^\pm(\vec k) = \sum_s v_a^{s\pm}(\vec k)\, a_s^\pm(\vec k) \quad \text{classically}$$

$$\hat\phi_a^-(\vec k) = \sum_s v_a^{-s}(\vec k)\, a_s(\vec k)$$

$$\hat\phi_a^+(\vec k) = \sum_s v_a^{-s+}(\vec k)\, b_s^+(\vec k)$$

Let us observe that

$$[A,BC] = ABC - BCA = ABC - BAC + \underbrace{BAC - BCA} = B[A,C] + [A,B]C$$
$$= [A,B]C - B[C,A]$$

$$[A,BC] = ABC - BCA = ABC + BAC - BAC - BCA =$$
$$= \{A,B\}C - B\{A,C\}$$

i.e. in both cases $[A,BC] = \{A,B\}_\pm C - B\{C,A\}_\pm$

Now from $k_\mu \hat\phi^-(\vec k) = +[\hat\phi^-(\vec k), P_\mu]$ we have

$$k_\mu \sum_s v_s^-(\vec k)\, a_s(\vec k) = \sum_{s,s'} v_s^-(\vec k) \int q_\mu \left[a_s(\vec k),\, a_{s'}^+(\vec q)\, a_{s'}(\vec q) \pm b_{s'}(\vec q)\, b_{s'}^+(\vec q)\right] d^3q$$

or

$$k_\mu a_s(\vec k) = \int d^3q\; q_\mu \sum_{s'} \{a_s(\vec k), a_{s'}^+(\vec q)\}\, a_{s'}(\vec q)$$

While from $k_\mu \hat\phi^+(\vec k) = -\left[\hat\phi^+(\vec k), \int \sum_{s'} q_\mu \left(a_{s'}^+(\vec q) a_{s'}(\vec q) \pm b_{s'}(\vec q) b_{s'}^+(\vec q)\right) d^3q\right]$

$$\mp k_\mu b_s^+(\vec{k}) = \int d^3q\, q_\mu \sum_{s'} \{ b_s^+(\vec{k}),\, b_{s'}(\vec{q}) \}\, b_{s'}^+(\vec{q})$$

(10Q)

Therefore

$$\{ a_s(\vec{k}),\, a_{s'}^+(\vec{q}) \} = \delta_{ss'}\, \delta(\vec{k}-\vec{q}) \qquad (\mathrm{I})$$

$$\{ b_s^+(\vec{k}),\, b_{s'}(\vec{q}) \} = \mp \delta_{ss'}\, \delta(\vec{k}-\vec{q}) \qquad (\mathrm{II})$$

The indefinite sign results from ± in the expression for P_μ

In conclusion at this stage for fields of each type we have obtained two kinds of commutation relations

THE REQUIREMENT THAT THESE RELATIONS SHOULD BE SYMMETRIC WITH RESPECT TO THE SUBSTITUTION OF ANTIPARTICLES FOR PARTICLES UNIQUELY DEFINE THE RECIPE FOR QUANTIZATION IN EACH CASE:

$$\left.\begin{aligned} a_s(\vec{k}) &\longleftrightarrow b_s(\vec{k}) \\ a_s^+(\vec{k}) &\longleftrightarrow b_s^+(\vec{k}) \end{aligned}\right\} \quad \text{This transformations denote CHARGE CONJUGATION}$$

particle $\longleftrightarrow$ antiparticle

Indeed if we take the upper sign, i.e. the „−" the symmetry of relation (I) and (II) to charge conjugation implied that the symbol $\{,\}$ should be understood as COMMUTATOR

Analogously for „+" sign in the equation II charge symmetry requires that the symbol $\{,\}$ is an ANTICOMMUTATOR

Thus we arrived at the conclusion that fields of integral spin (+ sign in the energy-momentum four-vector) are subject to Bose-Einstein quantization

$$[a_s(\vec{k}), a_{s'}^+(\vec{q})] = [b_s(\vec{k}), b_{s'}^+(\vec{q})] = \delta_{ss'}\, \delta(\vec{k}-\vec{q})$$

The fields of half-integer spin are satisfying the Fermi-Dirac quantization

$$\{ a_s(\vec{k}), a_{s'}^+(\vec{q}) \} = \{ b_s(\vec{k}), b_{s'}^+(\vec{q}) \} = \delta_{ab}\, \delta(\vec{k}-\vec{q})$$

One can pick-up one point as physical vacuum: ex $\begin{cases} \varphi_{01} = v \\ \varphi_{02} = 0 \end{cases}$

$$\frac{\partial^2 V}{\partial \varphi_1^2} = \left(-\mu^2 + \lambda(\varphi_1^2 + \varphi_2^2)\right) + 2\lambda\varphi_1^2$$

$$\frac{\partial^2 V}{\partial \varphi_2^2} = \left(-\mu^2 + \lambda(\varphi_1^2 + \varphi_2^2)\right) + 2\lambda\varphi_2^2 \qquad \Rightarrow \quad m^2_{ab} = \begin{pmatrix} 2\lambda v^2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\frac{\partial^2 V}{\partial \varphi_1 \partial \varphi_2} = 2\lambda \varphi_1 \varphi_2$$

Therefore the fluctuations of $\varphi_1 - v$ correspond to a massive particle with mass $m^2 = 2\lambda v^2$

— — of $\varphi_2' = \varphi_2$ corresponds to a massless excitation called „Goldstone boson"

It is observed that using these new fields the Lagrangian can be rewritten as:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_1')^2 + \frac{1}{2}(\partial_\mu \varphi_2')^2 - \frac{1}{2}(2\lambda v^2)\varphi_1'^2 + \lambda v \varphi_1'(\varphi_1'^2 + \varphi_2'^2) - \frac{\lambda}{4}(\varphi_1'^2 + \varphi_2'^2)^2$$

„The symmetry of the Lagrangian has been broken by breaking the symmetry of the vacuum"

↓

a massless particle, Goldstone boson appeared

General formulation: another and more useful parametrization

$$\phi(x) = \frac{1}{\sqrt{2}} \rho(x)\, e^{i\theta(x)/v} \qquad \text{with} \quad \begin{array}{l} v - \text{a constant} \\ \rho(x) \text{ and } \theta(x) \text{ real fields} \end{array}$$

Then

$$\partial_\mu \phi = \frac{1}{\sqrt{2}} e^{i\theta/v}\left(\partial_\mu \rho + \frac{i}{v}\rho\, \partial_\mu \theta\right)$$

and

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \rho)(\partial^\mu \rho) + \frac{1}{2v^2}\rho^2(\partial_\mu \theta)^2 - V(\rho^2)$$

To find the mass of the particle corresponding to the excitation of the would-be radial field $\rho(x)$ we expand it as $\rho(x) = v + \eta(x)$ and obtain

$$\mathcal{L} = \frac{1}{2}\partial_\mu \eta\, \partial^\mu \eta + \frac{1}{2}\partial_\mu \theta\, \partial^\mu \theta + \frac{1}{v}\eta\, \partial_\mu \theta\, \partial^\mu \theta + \frac{1}{2v^2}\eta^2 \partial_\mu \theta\, \partial^\mu \theta - V(\rho^2)$$

with $V(\rho) = \frac{1}{2}(2\mu^2)\eta^2 + \lambda v \eta^3 + \frac{\lambda}{4}\eta^4 - \frac{1}{4}\mu^2 v^2$; $\quad m_\eta = \sqrt{2\mu^2}$

It is noted that there is no quadratic term of θ in this Lagrangian. From this Lagrangian one identifies a massive η field with mass $m_\eta = \sqrt{2\mu^2}$ and a massless Goldstone boson θ

Extension to the SU(2) model

Now the model consider a field given by a doublet $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ composed of 2 complex fields

$$\begin{cases} \phi_1 = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) \\ \phi_2 = \frac{1}{\sqrt{2}}(\varphi_3 + i\varphi_4) \end{cases}$$

A SU(2) invariant Lagrangian is given.

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi^\dagger \phi) \quad \text{with} \quad V(\phi^\dagger\phi) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger\phi)^2$$

with $\mu^2 > 0$, $\lambda > 0$ We can use four real fields $H(x)$ and $\xi^i(x)$ with $i = 1,2,3$ to write ϕ as

$$\phi = \frac{1}{\sqrt{2}} e^{i\tau_i \xi^i(x)/2v} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}$$

where the VEV v is defined as $\varphi_{0_1}^2 + \varphi_{0_2}^2 + \varphi_{0_3}^2 + \varphi_{0_4}^2 \equiv v^2 = \frac{\mu^2}{\lambda}$

(or $(\phi^\dagger\phi)_0 \equiv |\phi_0|^2 = \frac{v^2}{2} = \frac{\mu^2}{2\lambda}$

Considering the expression

$$\partial_\mu \phi = \frac{1}{\sqrt{2}} e^{i\tau^i \xi^i/2v} \left\{ \begin{pmatrix} 0 \\ \partial_\mu H \end{pmatrix} + \frac{i}{v} \frac{\tau^i}{2} \partial_\mu \xi^i \begin{pmatrix} 0 \\ v + H \end{pmatrix} \right\}$$

into the Lagrangian density is obtained

$$\mathcal{L} = \frac{1}{2} \left\{ (0 \;\; \partial_\mu H) - \frac{i}{v} \partial_\mu \xi^i (0 \;\; v + H) \frac{\tau^i}{2} \right\} \left\{ \begin{pmatrix} 0 \\ \partial^\mu H \end{pmatrix} + \frac{i}{v} \frac{\tau^j}{2} \partial^\mu \xi^j \begin{pmatrix} 0 \\ v + H \end{pmatrix} \right\} -$$

$$- V((v + H)^2) \Rightarrow$$

$$\mathcal{L} = \frac{1}{2} \partial_\mu H \partial^\mu H + \frac{1}{8v^2} \partial_\mu \xi^i \partial^\mu \xi^i (v + H)^2 - V((v + H)^2)$$

using the relation $\tau^i \tau^j = \delta^{ij} + i\varepsilon_{ijk} \tau^k$

From this structure of this Lagrangian is observed that

- 3 fields ξ^i ($i = 1,3$) have no mass terms $\rightarrow$ massless Goldstone bosons
- $H(x)$ is now a massive field with mass $\sqrt{2\mu^2}$

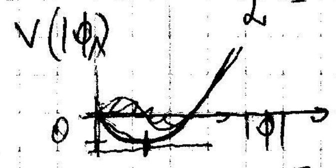
The number of Goldstone bosons is equal to the number of generators breaking the symmetry of the vacuum state

HIGGS MECHANISM

* extension to the situation when the spontaneous symmetry breaking is associated to local gauge symmetry:

The U(1) model

* scalar electrodynamics for a scalar complex field $\phi = \varphi_1 + i\varphi_2$



$$\mathcal{L} = (D_\mu \phi)^* (D^\mu \phi) - V(\phi^* \phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

with $V(\phi^*\phi) = -\mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2$ with $\mu^2 > 0$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ → field strength tensor for foton field A_μ

$D_\mu \phi = (\partial_\mu - i e A_\mu)\phi$ → the covariant derivative

e → "charge"

* the Lagrangian is invariant under U(1) local gauge transformations:

$$\phi \to \phi' = e^{-i\alpha(x)} \phi$$

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \alpha(x)$$

* The potential is minimum for $|\phi_0|^2 \equiv \frac{v^2}{2} = \frac{\mu^2}{2\lambda}$; $v = \sqrt{\frac{\mu^2}{\lambda}}$

* Let us parametrise the complex field $\phi(x)$ as

VERY IMPORTANT

$$\phi(x) = \frac{1}{\sqrt{2}} \left(v + \eta(x) \right) e^{i\theta(x)/v}$$

v position of the minimum

where now $\eta(x)$ and $\theta(x)$ are real fields. we can define a new set of fields by taking a particular gauge transformation with

UNITARY* GAUGE

$\alpha(x) = \frac{\theta(x)}{v}$, named "unitary gauge"

$$\phi(x) \to \phi'(x) = e^{-i\theta(x)/v} \phi(x) = \frac{1}{\sqrt{2}} \left(v + \eta(x) \right)$$

$$A_\mu(x) \to B_\mu(x) = A_\mu(x) - \frac{1}{ev} \partial_\mu \theta(x)$$

Under this gauge transformation:

$$D_\mu \phi(x) \to D'_\mu \phi'(x) = (\partial_\mu - ie B_\mu) \frac{1}{\sqrt{2}} (v + \eta(x))$$

$F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu \to F_{\mu\nu}(B) = \partial_\mu B_\nu - \partial_\nu B_\mu$ and

fields strength tensor $F_{\mu\nu}$ is gauge invariant (as it should be)

In terms of these "gauged" quantities the lagrangian is

$$\mathcal{L} = \frac{1}{2}|\partial_\mu \eta - ie B_\mu(v+\eta)|^2 + \frac{\mu^2}{2}(v+\eta)^2 - \frac{\lambda}{4}(v+\eta)^4 - \frac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B)$$

$$= \underbrace{\frac{1}{2}\partial_\mu \eta\, \partial^\mu \eta}\ + \underbrace{\mu^2 \eta^2 \mp \lambda v \eta^3 - \frac{\lambda}{4}\eta^4}$$

$$\underline{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}} + \underline{\frac{1}{2}(ev)^2 B_\mu B^\mu} + \underline{\underline{\frac{1}{2}e^2 B_\mu B^\mu \eta(\eta+2v)}}$$

This Lagrangian describes a massive vector boson B with mass

$$m_B = ev$$

and a scalar massive η with mass

$$m_\eta = \sqrt{2\mu^2}$$

named „Higgs boson". Here we have no Goldstone boson which has gone out from the Lagrangian

In summary, by extending the symmetry of the Lagrangian from the global to local one we found that the massless Goldstone boson θ disappeared and a massive gauge vector boson B and a massive scalar boson η, Higgs boson, came out

. The Goldstone boson θ was eaten up by the the gauge boson B (and become the longitudinal component of it)

We note that in the Higgs mechanism the degree of freedom is conserved, that is to say, starting from <u>two real scalar fields</u> (φ_1, φ_2) or (η, θ) plus <u>two polarisation states of massless photons A_μ</u> we finally got one real massive scalar field η and one massive vector boson B_μ with three polarisation degrees of freedom

Non-abelian SU(2) model

Again consider the doublet field $\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ and the gauge invariant Lagrangian with SU(2) symmetry is given by:

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - \frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} - V(\phi^\dagger\phi)$$

with

$$D_\mu\phi = \left(\partial_\mu - ig\frac{\tau^i}{2}A^i_\mu\right)\phi \qquad i=1,2,3$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\,\varepsilon_{ijk}A^j_\mu A^k_\nu$$

$$V(\phi^\dagger\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \quad \text{with } \mu^2>0$$

* Introduce new real fields $H(x)$ and $\xi^i(x)$ by parametrizing ϕ as follows:

$$\phi(x) = \frac{1}{\sqrt{2}}\, e^{i\tau^i\xi^i(x)/2v}\begin{pmatrix}0\\ v+H(x)\end{pmatrix}$$

* Take the unitary gauge with $U(x) = e^{-i\tau^i\xi^i(x)/2v}$ and define new fields as:

$$\phi(x) \to \phi'(x) = U(x)\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+H(x)\end{pmatrix}$$

$$\mathcal{A}_\mu \to \mathcal{B}_\mu = U(x)\mathcal{A}_\mu U^{-1}(x) - \frac{i}{g}(\partial_\mu U)U^{-1}$$

This transformation leads to:

$$D_\mu\phi \to (D_\mu\phi)' = \left(\partial_\mu - ig\frac{\tau^i}{2}B^i_\mu\right)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+H(x)\end{pmatrix}$$

$$F^i_{\mu\nu}(A)F^{i\mu\nu}(A) \to F^i_{\mu\nu}(B)F^{i\mu\nu}(B) = F^i_{\mu\nu}(A)F^{i\mu\nu}(A) \quad \text{with}$$

$$F^i_{\mu\nu}(B) = \partial_\mu B^i_\nu - \partial_\nu B^i_\mu + g\,\varepsilon_{ijk}B^j_\mu B^k_\nu$$

The Lagrangian becomes:

$$\mathcal{L} = (D_\mu\phi)'(D^\mu\phi)' - \frac{1}{4}F^i_{\mu\nu}(B)F^{i\mu\nu}(B) + \mu^2(\phi'^\dagger\phi') - \lambda(\phi'^\dagger\phi')^2$$

and is observed that the three fields $\xi_i(x)$ $i=1,2,3$ disappear.

Where did these fields (degrees of freedom) go?

Observed that:

$$\left[(D_\mu \phi)' \right]^{\dagger a} (D^\mu \phi)'_a = \frac{1}{2} \partial_\mu H \partial^\mu H + g^2 B^i_\mu B^{j\mu} \left(\frac{\tau^i}{2} \right)^a_b \left(\frac{\tau^j}{2} \right)^c_a \phi'^{b} \phi'_c$$

$$= \frac{1}{2} \partial_\mu H \partial^\mu H + \frac{g^2}{8} B^i_\mu B^{i\mu} (v + H)^2$$

So finally we obtain the following Lagrangian:

$$\mathcal{L} = \frac{1}{2} \partial_\mu H \partial^\mu H - \mu^2 H^2 - \lambda v H^3 - \frac{\lambda}{4} H^4 - \frac{v^4}{4}$$

$$- \frac{1}{4} F^i_{\mu\nu}(B) F^{i\mu\nu}(B) + \frac{g^2 v^2}{8} B^i_\mu B^{i\mu} + \frac{g^2}{8} B^i_\mu B^{i\mu} H(2v + H)$$

Conclusion: a triplet of massive vector fields B^i_μ $(i = 1,2,3)$ with mass

$$m_B = \frac{1}{2} g v$$

and a single massive scalar, i.e. Higgs boson with the mass

$$m_H = \sqrt{2\mu^2}$$

appeared. Is found again that the Goldstone bosons ξ^i $(i = \overline{1,3})$ were eaten by the gauge bosons B^i to make their longitudinal components. This is the Higgs mechanism in the non-Abelian $SU(2)$ gauge theory. The number of degrees of freedom is again conserved:

3 Goldstone bosons $\longrightarrow$ longitudinal components of the respective gauge fields which lead to appearance of 3 massive vector bosons B^i

LOCAL GAUGE SYMMETRIES

« All fundamental interactions are invariant under local gauge transformat.»

EM INTERACTION – is unbroken local $U(1)$ symmetric

WEAK INTERACTION – is spontaneously broken local $SU(2)\times U(1)$ symmetric

STRONG INTERACTION – is unbroken local $SU(3)$ symmetric

QED → U(1) model

Work with $\hbar = c = 1$;

$\mathcal{L}(x) = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)$ → the lagrangian for free fermionic field with the mass m

$\psi(x)$ satisfy Dirac eq. $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$

Global transf:

$$\psi'(x) = e^{-i\theta}\psi(x) = U_\theta\psi(x)$$

U_θ is a phase transf., the same at all space-time points

$$U_\theta^\dagger U_\theta = I \;;\; U_{\theta_1}U_{\theta_2} = U_{\theta_2}U_{\theta_1} \rightarrow \text{ABELIAN GROUP}$$

$$U_\theta \in U(1)$$

Local transf: now $\theta(x)$ is varying from point to point in space-time

$$\psi'(x) = e^{-i\theta(x)}\psi(x)$$
$$\bar{\psi}'(x) = e^{i\theta(x)}\bar{\psi}(x)$$

$$\mathcal{L} \rightarrow \mathcal{L}' = \bar{\psi}'(x)(i\gamma^\mu\partial_\mu - m)\psi'(x) =$$
$$= \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) + \bar{\psi}(x)\gamma^\mu\psi(x)\partial_\mu\theta(x) = \mathcal{L} + j^\mu\partial_\mu\theta(x)$$

with $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$ the vector current associated to the fermionic field. Is observed that the Lagrangian of the free fermionic field is not invariant under the local transformations. If the following replacement is performed: $\partial_\mu \rightarrow D_\mu \equiv \partial_\mu - ieA_\mu(x)$ the new Lagrangian is $\mathcal{L}_i = \mathcal{L} + ej^\mu A_\mu$ where here $A_\mu(x)$ is an additional vector field which is included in theory

$$\mathcal{L}_i = \mathcal{L}(\psi, \bar{\psi}, A_\mu) = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + e\bar{\psi}\gamma^\mu\psi A_\mu$$

Then

$$\mathcal{L}_i' = \mathcal{L} + j^\mu \partial_\mu \theta(x) + e j^\mu A_\mu'$$

and $\mathcal{L}_i' = \mathcal{L}_i$ only if $j^\mu \partial_\mu \theta(x) + e j^\mu A_\mu' = e j^\mu A_\mu$

i.e. $j^\mu(x)\left(\partial_\mu \theta(x) + eA_\mu'(x) - eA_\mu(x)\right) = 0$

$$\Rightarrow A_\mu \to A_\mu' = A_\mu - \frac{1}{e}\partial_\mu \theta(x)$$

Conclusion: the electromagnetic dynamics of a charged fermion field is made invariant to the local gauge transformations by introducing a spin 1 vector (gauge) field A_μ called the photon, through the covariant derivative, called also "minimal coupling"

Step 2: add Lagrangian for the vector field

$\frac{1}{2} m^2 A_\mu A^\mu$ → violates the local gauge invariance if $m \neq 0$

Introducing $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the $\mathcal{L}_{photon} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

- This term (kinetic term) is invariant to the local gauge transf for A_μ as easily checked by direct replacement

Alternatively is observed that $[D_\mu, D_\nu]\Psi = -ieF_{\mu\nu}\Psi$ and then from $\left([D_\mu, D_\nu]\Psi\right)' = e^{-i\theta(x)}\left([D_\mu, D_\nu]\Psi\right)$ is concluded that

$$F_{\mu\nu}'\Psi' = e^{-i\theta(x)} F_{\mu\nu}\Psi = F_{\mu\nu}\Psi' \text{ that is } F'^{\mu\nu} = F^{\mu\nu}$$

$$\mathcal{L}_{QED} = \bar{\Psi}(x)\left(i\gamma^\mu D_\mu - m\right)\Psi(x) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

YANG - MILLS GAUGE THEORY - SU(2) MODEL

" Under what conditions is a theory invariant under a space-time and isospin dependent phase transformation?

SU(2) Gauge transf. of the fermionic field:

$$\Psi(x) \longrightarrow \Psi'(x) = U(x)\,\Psi(x) \quad \text{where}$$

$$U(x) = \exp\left(-ig\frac{\tau^i}{2}\theta^i(x)\right)$$

$\tau^i \to$ the Pauli matrices $i=1,2,3$ $(3=2^2-1)$ θ^i real parameters

$g \to$ the coupling constant

$$\left[\frac{\tau^i}{2}, \frac{\tau^j}{2}\right] = i\,\varepsilon_{ijk}\frac{\tau^k}{2} \quad ; \quad \varepsilon_{ijk} \to \text{totally anti-symmetric Levy-Civita symbol}$$

$\downarrow$ Structure constants of SU(2)

Then Ψ is a two-component spinor i.e. is a dublet Dirac field

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \quad ; \quad \bar{\Psi} = \begin{pmatrix} \bar{\Psi}_1 & \bar{\Psi}_2 \end{pmatrix} \quad ; \quad \bar{\Psi}_a = \Psi_a^{+}\gamma^0 \text{ with } a=1,2$$

ex: $\Psi = \begin{pmatrix} u \\ d \end{pmatrix}$ where u & d represents up and down quark fields

Now $A^i_\mu(x)$ represents the new gauge fields, three independent components associated to the isospin index

$$\vec{A}_\mu = \sum_{i=1}^{3} \frac{\tau^i}{2} A^i_\mu = \frac{\vec{\tau}}{2}\cdot\vec{A}_\mu$$

The covariant derivative

$$D_\mu = \partial_\mu - ig\vec{A}_\mu$$

should satisfy $(D_\mu\Psi)' = D'_\mu\Psi' = U(D_\mu\Psi)$ so $D'_\mu = UD_\mu U^{-1}$

If we demand for $\vec{A}_\mu$ to transform as

$$\vec{A}'_\mu = U\vec{A}_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}$$

is observed that

$$(D_\mu\Psi)' = (\partial_\mu - ig\vec{A}'_\mu)U\Psi = (\partial_\mu U)\Psi + U\partial_\mu\Psi - ig\left[U\vec{A}_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}\right]U\Psi$$

$$= U(\partial_\mu\Psi - ig\vec{A}_\mu\Psi) = U(D_\mu\Psi)$$

i.e. the required rule is verified

Considering infinitesimal transformations

$$\begin{cases} U \simeq I - i \sum_{j=1}^{3} \frac{\tau_j}{2} \theta^j = I - i \frac{\vec{\tau}}{2} \cdot \vec{\theta} \equiv I - i\theta \\ U^{-1} \simeq I + i \frac{\vec{\tau}}{2} \cdot \vec{\theta} \equiv I + i\theta \end{cases}$$

and the transformation rule for gauge fields is

$$\vec{A}'_\mu = U \vec{A}_\mu U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1} = \left(I - i \frac{\vec{\tau}}{2} \cdot \vec{\theta}\right) \vec{A}_\mu \left(1 + i \frac{\vec{\tau}}{2} \cdot \vec{\theta}\right)$$

$$- \frac{i}{g} \left(-i \frac{\vec{\tau}}{2} \cdot \partial_\mu \vec{\theta}\right) \left(I + i \frac{\vec{\tau}}{2} \vec{\theta}\right) =$$

$$= \vec{A}_\mu - i [\theta, \vec{A}_\mu] - \frac{1}{g} \partial_\mu \theta$$

$$A^{i\prime}_\mu = A^i_\mu + \varepsilon_{ijk} \theta^j A^k_\mu - \frac{1}{g} \partial_\mu \theta^i \ ; \quad \delta A^i_\mu = \varepsilon_{ijk} \theta^j A^k_\mu - \frac{1}{g} \partial_\mu \theta^i$$

Generalize the relation $[D_\mu, D_\nu]\psi = -ie F_{\mu\nu} \psi$ as

$[D_\mu, D_\nu]\psi = -ig \vec{F}_{\mu\nu} \psi$ where $\vec{F}_{\mu\nu} = \sum_{i=1}^{3} \frac{\tau^i}{2} F^i_{\mu\nu} = \frac{\vec{\tau}}{2} \cdot \vec{F}_{\mu\nu}$

defined by $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g \varepsilon_{ijk} A^j_\mu A^k_\nu$

Proof: $[D_\mu, D_\nu]\psi = [\partial_\mu - ig\vec{A}_\mu, \partial_\nu - ig\vec{A}_\nu]\psi =$

$$= (\partial_\mu - ig\vec{A}_\mu)(\partial_\nu - ig\vec{A}_\nu)\psi - (\partial_\nu - ig\vec{A}_\nu)(\partial_\mu - ig\vec{A}_\mu)\psi =$$

$$= \cancel{\partial_\mu \partial_\nu \psi} - \cancel{ig\vec{A}_\mu \partial_\nu \psi} - ig(\partial_\mu \vec{A}_\nu)\psi - \cancel{ig\vec{A}_\nu \partial_\mu \psi} - g^2 \vec{A}_\mu \vec{A}_\nu \psi$$

$$\cancel{-\partial_\nu \partial_\mu \psi} + \cancel{ig\vec{A}_\nu \partial_\mu \psi} + ig(\partial_\nu \vec{A}_\mu)\psi + \cancel{ig\vec{A}_\mu \partial_\nu \psi} + g^2 \vec{A}_\nu \vec{A}_\mu \psi$$

$$= -ig\left(\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu - ig(\vec{A}_\mu \cdot \vec{A}_\nu - \vec{A}_\nu \cdot \vec{A}_\mu)\right)\psi$$

$$\vec{A}_\mu \cdot \vec{A}_\nu - \vec{A}_\nu \cdot \vec{A}_\mu = \sum_{i,j} \left(\frac{\tau_i}{2} A^i_\mu \frac{\tau_j}{2} A^j_\nu - \frac{\tau_j}{2} \frac{\tau_i}{2} A^j_\nu A^i_\mu \right)$$

$$= \sum_{i,j} \left[\frac{\tau_i}{2}, \frac{\tau_j}{2}\right] A^i_\mu A^j_\nu = \sum_{i,j} i \varepsilon_{ijk} \frac{\tau_k}{2} A^j_\mu A^k_\nu$$

so

$$= -ig\left(\frac{\tau_i}{2} \partial_\mu A^i_\nu - \frac{\tau_i}{2} \partial_\nu A^i_\mu - i^2 g \frac{\tau_i}{2} \varepsilon_{ijk} A^j_\mu A^k_\nu\right)$$

$$= -ig \frac{\tau_i}{2} F^i_{\mu\nu}$$

Since $\left(D_\mu \Psi\right)' = U D_\mu \Psi$

$$\left([D_\mu, D_\nu]\Psi\right)' = \left(D_\mu D_\nu \Psi\right)' - \left(D_\nu D_\mu \Psi\right)' = D'_\mu U D_\nu \Psi - D'_\nu U D_\mu \Psi =$$

$$= U\left[\left(D_\mu U^{-1} U D_\nu\right) - U\left(D_\nu U^{-1} U D_\mu\right)\right]\Psi = U\left([D_\mu, D_\nu]\Psi\right)$$

one derives that

$$\vec{\mathcal{F}}'_{\mu\nu}\Psi' = U\mathcal{F}_{\mu\nu}\Psi \quad \text{or} \quad \vec{\mathcal{F}}'_{\mu\nu} U = U\mathcal{F}_{\mu\nu} \Rightarrow$$

$$\vec{\mathcal{F}}'_{\mu\nu} = U\vec{\mathcal{F}}_{\mu\nu}U^{-1} \Rightarrow \delta\mathcal{F}^i_{\mu\nu} = \varepsilon_{ijk}\theta^j\mathcal{F}^k_{\mu\nu}$$

This shows that $F^i_{\mu\nu}$ transforms nontrivially unlike the case of QED where $F_{\mu\nu}$ is invariant because of no structure constant in an abelian group

The kinematic term associated to gauge field is

$$-\frac{1}{2}\mathrm{Tr}\left(\vec{\mathcal{F}}_{\mu\nu}\cdot\vec{\mathcal{F}}_{\mu\nu}\right) = -\frac{1}{2}\sum_{i,j}\mathrm{Tr}\left(\frac{\tau_i}{2}F^i_{\mu\nu}\frac{\tau_j}{2}F^{j\mu\nu}\right) = -\frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu}$$

$$\text{since} \quad \mathrm{Tr}\,\frac{\tau_i}{2}\frac{\tau_j}{2} = \frac{1}{2}\delta^{ij}$$

This term is indeed invariant under the gauge transformations

$$\delta\left(F^i_{\mu\nu}F^{i\mu\nu}\right) = 2\delta F^i_{\mu\nu}F^{i\mu\nu} = 2\underset{\text{antisym}}{\varepsilon_{ijk}}\,\theta^j\underbrace{F^k_{\mu\nu}F^{i\mu\nu}}_{\text{symm}} = 0$$

SUMMARY: We can write down the gauge invariant Lagrangian of a fermionic field with mass m in an $SU(2)$ symmetric world:

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_G$$

$$\mathcal{L}_F = \bar{\Psi}\left(i\gamma^\mu D_\mu - m\right)\Psi$$

$$\mathcal{L}_G = -\frac{1}{2}\mathrm{Tr}\left(\tilde{\mathcal{F}}_{\mu\nu}\cdot\tilde{\mathcal{F}}^{\mu\nu}\right) = -\frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu}$$

$$D_\mu = \partial_\mu - ig\frac{\tau_i}{2}A^i_\mu$$ → contains three gauge fields A^i_μ

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\varepsilon_{ijk}A^j_\mu A^k_\nu$$ → the gauge field tensor

Comment 1: Contrary to QED the kinetic term of the gauge fields $\mathcal{L}_G$, contains 3 gauge boson interactions with the same coupling constants g as the one of the gauge fields to fermions. It contains also a 4 gauge boson interaction with the coupling g^2:

$$\mathcal{L}_G = -\frac{1}{4}\left(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\varepsilon_{ijk}A^j_\mu A^k_\nu\right)\left(\partial^\mu A^{i\nu} - \partial^\nu A^{i\mu} + g\varepsilon_{ilm}A^{l\mu}A^{m\nu}\right)$$

$$= -\frac{1}{2}\partial_\mu A^i_\nu\left(\partial^\mu A^{i\nu} - \partial^\nu A^{i\mu}\right) - g\varepsilon_{ijk}A^i_\mu A^j_\nu \partial^\mu A^{k\nu} - \frac{g^2}{4}\varepsilon_{ijk}\varepsilon_{ilm}A^j_\mu A^k_\nu A^{l\mu}A^{m\nu}$$

Therefore Y-M is not a free theory even without matter fields because it contains self-interactions among gauge fields; this is very different from Abelian gauge theory like QED where there is no self-coupling of photons

Comment 2: As long as we demand the gauge invariance alone we can add renormalisable Yukawa interactions such as

$$\mathcal{L}_Y = G_Y \bar{\psi}(x)\psi(x)\phi(x)$$

this is invariant under gauge transf.

Comment 3 this theory is not useful for weak interaction because it gives identical couplings to right and left-handed fermions and leads to the parity conservation;

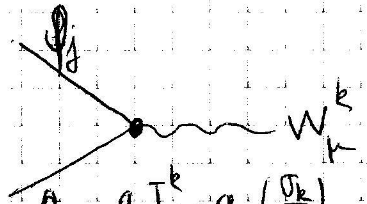
To preserve gauge invariance, it is essential to have massless gauge bosons A^i_μ which should give weak interactions of infinite range like electromagnetic case, contrary to experimental observations

SU(2)		SU(3)
$\psi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$	$\psi \to \psi' = U\psi$ $U = \exp\left(i\bar{\theta}(x)\cdot\bar{T}\right)$	$\psi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$
$\bar{\theta}\cdot\bar{T} = \theta^1 T^1 + \theta^2 T^2 + \theta^3 T^3$	$[T^j, T^k] = i c^{jkm} T^m$	$\bar{\alpha}\cdot\bar{T} = \alpha^1 T^1 + \dots \alpha^8 T^8$
$T^j = \frac{\sigma^j}{2}$	$\mathrm{Tr}[T^j T^k] = \frac{\delta_{jk}}{2}$	$T^j = \frac{\lambda^j}{2} \quad j = \overline{1,8}$
$c_{jkm} = \varepsilon_{jkm}$	$D_\mu \to D'_\mu = U D_\mu U^{-1}$	$c_{jkm} = f^{jkm}$
$D_\mu = \partial_\mu - i g_2 \frac{\bar{\sigma}\cdot\bar{W}_\mu}{2}$	$D_\mu = \partial_\mu - i g \bar{T}\cdot\bar{V}_\mu$	$D_\mu = \partial_\mu - i g_3 \frac{\bar{\lambda}\cdot\bar{G}_\mu}{2}$

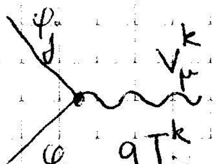
$$\mathcal{L}_0 = i\bar{\psi}\gamma^\mu\partial_\mu\psi \to i\bar{\psi}\gamma^\mu D_\mu\psi$$

$$\mathcal{L}_{int} = g\bar{\psi}\gamma^\mu\bar{T}\cdot\bar{V}_\mu\psi$$

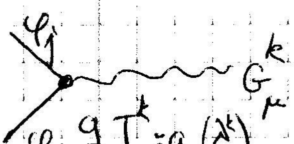
$$\mathcal{L}'_{int} = g_2(\bar{\varphi}_1\ \bar{\varphi}_2)\begin{pmatrix} W^3_\mu & W^1_\mu - iW^2_\mu \\ W^1_\mu + iW^2_\mu & -W^3_\mu \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} ; \quad \mathcal{L}_{int} = g_3(\bar{\varphi}_1\ \bar{\varphi}_2\ \bar{\varphi}_3)\begin{pmatrix} G^3_\mu + \frac{G^8_\mu}{\sqrt{3}} & G^1_\mu - iG^2_\mu & G^4_\mu - iG^5_\mu \\ G^1_\mu + iG^2_\mu & -G^3_\mu + \frac{G^8_\mu}{\sqrt{3}} & G^6_\mu - iG^7_\mu \\ G^4_\mu + iG^5_\mu & G^6_\mu + iG^7_\mu & -\frac{\sqrt{2}G^8_\mu}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$



$g_2 T^k_{ji} = g_2\left(\frac{\sigma_k}{2}\right)_{ji}$



$g T^k_{ji}$



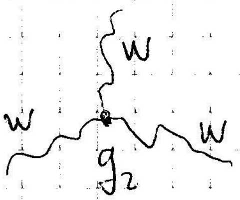
$g_3 T^k_{ji} = g_3\left(\frac{\lambda^k}{2}\right)_{ji}$

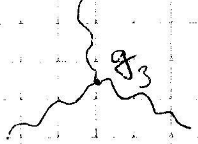
$$\mathcal{L}(V) = -\frac{1}{2}\mathrm{Tr}(V_{\mu\nu}V^{\mu\nu}) = -\frac{1}{4}\sum V^j_{\mu\nu}V^{j\mu\nu}$$

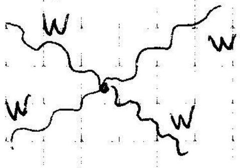
$$W_{\mu\nu} = \frac{\bar{\sigma}}{2}\cdot\bar{W}_{\mu\nu} \qquad V_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \bar{T}\cdot\bar{V}_{\mu\nu} \qquad G_{\mu\nu} = \frac{\bar{\lambda}}{2}\cdot\bar{G}_{\mu\nu}$$

$$W^j_{\mu\nu} = \partial_\mu W^j_\nu - \partial_\nu W^j_\mu + g_2\varepsilon^{jkm}W^k_\mu W^m_\nu$$

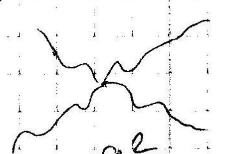
$$G^i_{\mu\nu} = \partial_\mu G^i_\nu - \partial_\nu G^i_\mu + g_3 f^{ikm}G^k_\mu G^m_\nu$$







g_2^2



g_3^2

SPONTANEOUS SYMMETRY BREAKING

GOLDSTONE THEOREM:

Let be a system described by a complex scalar field $\phi(x)$ and $\phi^*(x)$ with Lagrangian density

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi^*\phi) \quad \text{with} \quad V(\phi^*\phi) = m\phi^*\phi + \lambda(\phi^*\phi)^2$$

The system has a $U(1)$ global invariance. Introduce from

$$\phi = (\varphi_1 + i\varphi_2)/\sqrt{2} \quad ; \quad \phi^* = (\varphi_1 - i\varphi_2)/\sqrt{2}$$

two real field φ_1 and φ_2 with an equivalent Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 - V(\varphi_1^2 + \varphi_2^2)$$

This Lagrangian has an $O(2)$ symmetry, i.e. is invariant under the group transf. which belong to $O(2)$ group

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \to \begin{pmatrix}\varphi_1'\\ \varphi_2'\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

$V(\varphi_1^2+\varphi_2^2)$ → should be at most the 4-th order of the fields in order to ensure renormalizability of the theory

→ bounded below for change of $(\phi^*\phi)^{1/2} = |\phi|$ so that the theory has a stable ground state

Ex: $V = m(\varphi_1^2+\varphi_2^2) + \lambda(\varphi_1^2+\varphi_2^2)^2$ with $\lambda > 0$

In Quantum theory → particles are excitations of the field (quantized) about its lowest energy state, named VACUUM STATE

VEV - Vacuum expectation value of the field → the constant value of the field corresponding to the lowest energy state. $\phi_0 = \langle 0|\phi|0\rangle$

To find the particle spectra → expand the potential about its minimum corresponding to the lowest energy state as:

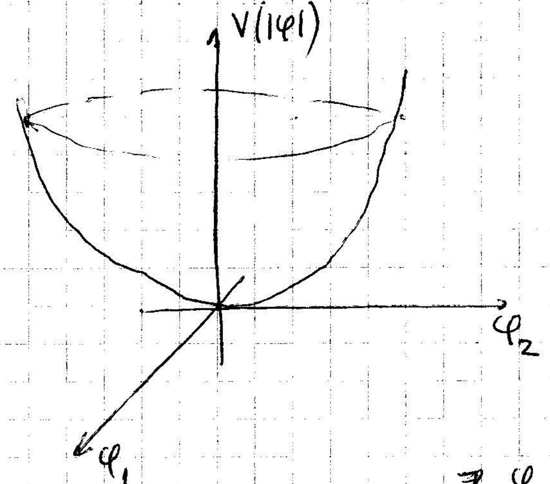
$$V(\varphi_1,\varphi_2) = V(\varphi_{01},\varphi_{02}) + \sum_{a=1,2}\left(\frac{\partial V}{\partial\varphi_a}\right)_0(\varphi_a - \varphi_{0a}) + \frac{1}{2}\sum_{a,b=1,2}\left(\frac{\partial^2 V}{\partial\varphi_a\partial\varphi_b}\right)_0(\varphi_a-\varphi_{0a})(\varphi_b-\varphi_{0b}) + \dots$$

$\phi_0 = (\varphi_{01},\varphi_{02})$ is the VEV of $\phi = (\varphi_1,\varphi_2)$

$V(\varphi_1, \varphi_2) = \min$ for $\phi_0 \Rightarrow$ the second term (associated to first order derivatives) cancels

$$\left(\frac{\partial^2 V}{\partial \varphi_a \partial \varphi_b}\right)_0 = m^2_{ab}$$ → mass matrix : by diagonalization generate the particle spectrum

CASE I → vacuum is unique → WIGNER PHASE (REALIZATION) with $m^2 > 0$; $\lambda > 0$



$$V(\varphi_1^2 + \varphi_2^2) = \frac{m^2}{2}(\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2$$

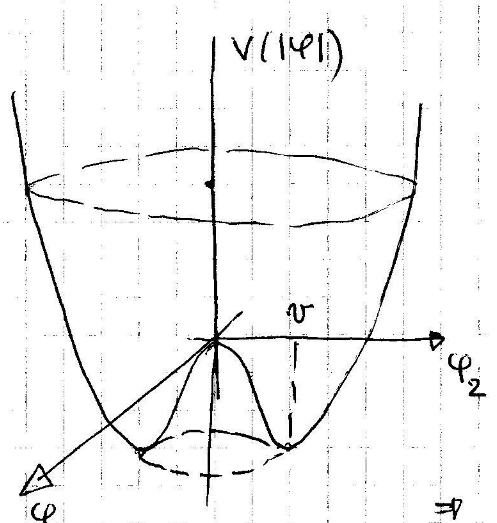
$$\begin{cases} \left(\dfrac{\partial V}{\partial \varphi_1}\right)_0 = m^2 \varphi_{01} + \lambda \varphi_{01}(\varphi_{01}^2 + \varphi_{02}^2) = 0 \\ \left(\dfrac{\partial V}{\partial \varphi_2}\right)_0 = m^2 \varphi_{02} + \lambda \varphi_{02}(\varphi_{01}^2 + \varphi_{02}^2) = 0 \end{cases}$$

$\Rightarrow \varphi_{01} = \varphi_{02} = 0$ → unique vacuum

The mass matrix becomes diagonal in this case: $m^2_{ab} = \begin{pmatrix} m^2 & 0 \\ 0 & m^2 \end{pmatrix}$

i.e. φ_1 and φ_2 have the same mass

CASE II → the vacuum is not unique - NAMBU-GOLDSTONE PHASE with $m_2 = -\mu^2 < 0$; $\lambda > 0$



$$V(\varphi_1^2 + \varphi_2^2) = -\frac{\mu^2}{2}(\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2$$

$$\begin{cases} \left(\dfrac{\partial V}{\partial \varphi_1}\right)_0 = 0 \Rightarrow -\mu^2 \varphi_{01} + \lambda \varphi_{01}(\varphi_{01}^2 + \varphi_{02}^2) = 0 \\ \left(\dfrac{\partial V}{\partial \varphi_2}\right)_0 = 0 \Rightarrow -\mu^2 \varphi_{02} + \lambda \varphi_{02}(\varphi_{01}^2 + \varphi_{02}^2) = 0 \end{cases}$$

$$\Rightarrow \varphi_{01}^2 + \varphi_{02}^2 \equiv v^2 = \frac{\mu^2}{\lambda} \quad \text{or} \quad |\phi_0|^2 = \frac{v^2}{2} = \frac{\mu^2}{2\lambda}$$

⇒ a continuously (infinitely) degenerate vacuum states with $\varphi_{01} \neq 0$ and/or $\varphi_{02} \neq 0$

All points on a circle with radius $v = \sqrt{\frac{\mu^2}{\lambda}}$ in the (φ_1, φ_2) plane correspond to the minimum of V.

SUBIECTE EXAMEN

TEORIA PARTICULELOR ELEMENTARE și INTRODUCERE ÎN TEORIA CÎMPURILOR CUANTICE

1. Ordine de mărime în fizica particulelor elementare
2. Sisteme de particule identice. Bosoni. Fermioni
3. Grupul Lorentz. Generatori. Relații de comutare
4. Grupul Poincare. Generatori. Relații de comutare
5. Reprezentări ireductibile finite ~~de~~ grupului Lorentz. Clasificarea cîmpurilor relativiste
6. Reprezentări unitare ale grupului Poincare. Stări uniparticulă
7. Cîmpul scalar real. Ecuația Klein-Gordon.
   Cîmpul scalar complex. Sarcina electrică. Elemente de cuantificare
8. Cîmpul vectorial cu $m \neq 0$. Ecuația Proca. El. de cuantificare
9. Cîmpul electromagnetic. Etalonări
10. Spinori. Cîmpul Dirac. Ecuația Dirac. Cazul $m \neq 0$.
    Cazul $m = 0$
    Elemente de cuantificare
11. Ruperea spontană de simetrie. Teorema Goldstone
12. Teorii de etalonare neabeliene. Cazul SU(2)
13. Teorii de etalonare neabeliene. Cazul SU(3)
14. Mecanismul Higgs.

1a) Write the Maxwell equations in vacuum and explain the physical meaning of the physical quantities

b) Starting with ME ~~sho~~ prove the electric charge conservation law

c) Show the existence of the em. wave. How are oriented the electric field intensity $\vec{E}$, magnetic induction and the wave vector $\vec{k}$ in a plane wave?

d) Express the energy density of the electromagnetic field, the energy current density, (Poynting vector) and enunciate the Poynting theorem

e) define the electromagnetic potentials.

2) Decomposing the electric/magnetic field and the ~~elec~~ current density into longitudinal ($\mathrm{div}\,\vec{v}=0$) and transversal components show that

$\vec{E}_{\parallel}$ is „instantaneous"

The transverse fields are determined by the transverse components of $\vec{J}$

3) Deduce the Fresnel relations in the case of an ~~incident~~ monochromatic plane wave with $\vec{E}_0 \perp (\vec{n}, \vec{k}_i)$

With $\eta^{\mu\nu} = \text{diag}(-1,1,1,1)$ we proved that

$$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} + \eta^{\mu\sigma}J^{\nu\rho} - \eta^{\nu\sigma}J^{\mu\rho}$$

With $g^{\mu\nu} = \text{diag}(1,-1,-1,-1) = -\eta^{\mu\nu}$ multiply by i both sides of the commutation relation:

Then:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left[g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right]$$

(as in Michele Maggiore pag 19)

Remember that $J^{\mu\nu} = -J^{\nu\mu}$

If $J^i = \frac{1}{2}\varepsilon^{ijk}J^{jk} \Rightarrow \varepsilon^{lmi}J^i = \frac{1}{2}\varepsilon^{lmi}\varepsilon^{jki}J^{jk} = \frac{1}{2}(\delta^{lj}\delta^{mk} - \delta^{lk}\delta^{mj})J^{jk} = J^{lm}$ Peskin pag 39

$K^i = J^{i0}$

then the Lie algebra of the Lorentz group becomes

$$[J^i, J^j] = i\varepsilon^{ijk}J^k$$
$$[J^i, K^j] = i\varepsilon^{ijk}K^k$$
$$[K^i, K^j] = -i\varepsilon^{ijk}J^k$$

Proof: $[J^i,J^j] = \frac{1}{4}\varepsilon^{ilm}\varepsilon^{jpq}[J^{lm},J^{pq}] = \frac{i}{4}\varepsilon^{ilm}\varepsilon^{jpq}\left(g^{mp}J^{lq} - g^{lp}J^{mq} - g^{mq}J^{lp} + g^{lq}J^{mp}\right)$

$= -\frac{i}{4}\varepsilon^{ilm}\varepsilon^{jpq}\left(\delta^{mp}J^{lq} - \delta^{lp}J^{mq} - \delta^{mq}J^{lp} + \delta^{lq}J^{mp}\right) =$

$= -\frac{i}{4}\varepsilon^{ilm}\varepsilon^{jmq}\varepsilon^{lqk}J^k \to +\frac{i}{4}(\delta^{ij}\delta^{lq} - \delta^{iq}\delta^{lj})\varepsilon^{lqk}J^k = -\frac{i}{4}\varepsilon^{jik}J^k = \frac{i}{4}\varepsilon^{ijk}J^k$

$+\frac{i}{4}\varepsilon^{ilm}\varepsilon^{jlq}\varepsilon^{mqk}J^k \to \frac{i}{4}(\delta^{ij}\delta^{mq} - \delta^{iq}\delta^{mj})\varepsilon^{mqk}J^k = -\frac{i}{4}\varepsilon^{jik}J^k = \frac{i}{4}\varepsilon^{ijk}J^k$

$+\frac{i}{4}\varepsilon^{ilm}\varepsilon^{jpm}\varepsilon^{lpk}J^k$

$-\frac{i}{4}\varepsilon^{ilm}\varepsilon^{jpl}\varepsilon^{mpk}J^k$

$= i\varepsilon^{ijk}J^k$ q.e.d.

$[J^i, K^j] = \frac{1}{2}\varepsilon^{ilm}[J^{lm}, J^{j0}] = \frac{i}{2}\varepsilon^{ilm}\left(g^{l0}J^{mj} + g^{mj}J^{l0} - g^{lj}J^{m0} - g^{m0}J^{lj}\right) =$

$= \frac{i}{2}\varepsilon^{ilm}(-)\delta^{mj}K^l - \frac{i}{2}\varepsilon^{ilm}(-)\delta^{lj}K^m = -\frac{i}{2}\varepsilon^{ilj}K^l + \frac{i}{2}\varepsilon^{ijm}K^m = +\frac{i}{2}\varepsilon^{ijk}K^k + \frac{i}{2}\varepsilon^{ijk}K^k$

$= +i\varepsilon^{ijk}K^k$ q.e.d.

$[K^i, K^j] = [J^{i0}, J^{j0}] = i\left(\delta^{i0}J^{0j} + \delta^{0j}J^{i0} - \delta^{ij}J^{00} - \delta^{00}J^{ij}\right) = -i\varepsilon^{ijk}J^k$ q.e.d.

Consider the transformation of coordinates

$$x \longrightarrow x'$$

nonsingular:

$x'(x)$ and $x(x')$ are well-behaved differentiable functions so that the matrix $\dfrac{\partial x'^{\alpha}}{\partial x^{\beta}}$ has a well defined inverse $\dfrac{\partial x^{\beta}}{\partial x'^{\alpha}}$

THEOREM: The Lorentz transformations are the only nonsingular coordinate transformations $x \to x'$ that leave invariant the proper time $d\tau$

$$d\tau^2 = dt^2 - d\vec{x}^{\,2} = -g_{\mu\nu}\, dx^{\mu} dx^{\nu}$$

Proof:

$$d\tau'^2 = -g_{\alpha\beta}\, dx'^{\alpha} dx'^{\beta} = -g_{\alpha\beta} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} dx^{\gamma} dx^{\delta}$$

$$d\tau^2 = -g_{\gamma\delta}\, dx^{\gamma} dx^{\delta}$$

$$\overset{\text{invariance of } d\tau}{\Longrightarrow} \boxed{g_{\alpha\beta} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} = g_{\gamma\delta}}$$

If we differentiate with respect to x^{ρ} $\Rightarrow$

$$0 = g_{\alpha\beta} \frac{\partial^2 x'^{\alpha}}{\partial x^{\gamma} \partial x^{\rho}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} + g_{\alpha\beta} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial^2 x'^{\beta}}{\partial x^{\delta} \partial x^{\rho}}$$

If instead of γ we have ρ and instead of ρ we put γ obtain

$$0 = g_{\alpha\beta} \frac{\partial^2 x'^{\alpha}}{\partial x^{\rho} \partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} + g_{\alpha\beta} \frac{\partial x'^{\alpha}}{\partial x^{\rho}} \frac{\partial^2 x'^{\beta}}{\partial x^{\delta} \partial x^{\gamma}}$$

Analogously if instead of δ we have ρ and instead of ρ we put δ we obtain

$$0 = g_{\alpha\beta} \frac{\partial^2 x'^{\alpha}}{\partial x^{\gamma} \partial x^{\delta}} \frac{\partial x'^{\beta}}{\partial x^{\rho}} + g_{\alpha\beta} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial^2 x'^{\beta}}{\partial x^{\rho} \partial x^{\delta}} \quad \Big|\cdot -1$$

$$\|$$

$$g_{\beta\alpha} \frac{\partial^2 x'^{\beta}}{\partial x^{\gamma} \partial x^{\delta}} \frac{\partial x'^{\alpha}}{\partial x^{\rho}} = g_{\alpha\beta} \frac{\partial^2 x'^{\beta}}{\partial x^{\gamma} \partial x^{\delta}} \frac{\partial x'^{\alpha}}{\partial x^{\rho}}$$

Summing up the last three equalities from the cancelations of various term results:

$$2 g_{\alpha\beta} \frac{\partial^2 x'^{\alpha}}{\partial x^{\gamma} \partial x^{\rho}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} = 0$$

For a given γ (fixed) this equation can be written as

$$[X^{(\gamma)}]_{\rho\alpha} \cdot [g]_{\alpha\beta} \cdot [M]_{\beta\delta} = 0 = [R]_{\rho\delta} \quad (\forall)\ \rho, \delta = \overline{0,3}$$