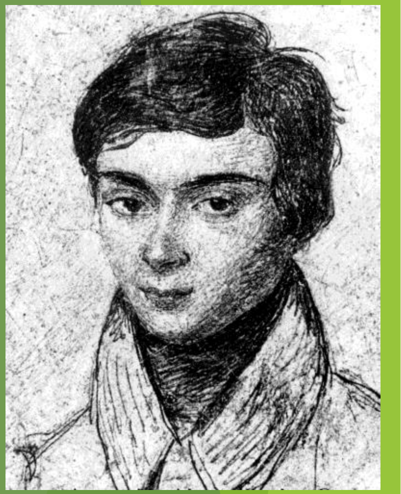
Group Theory

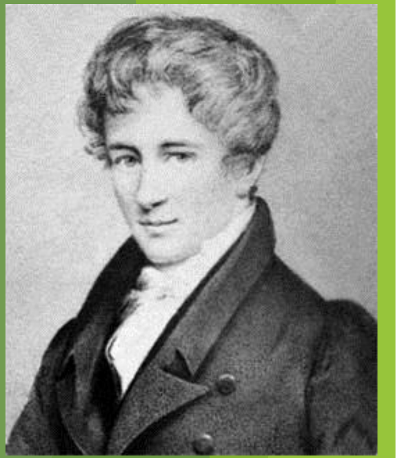
General Framework

Definition of a group $(G, \circ)$



- A set G endowed with an operation, $\circ : G \times G \to G$, called the *group law* of G, verifying 4 axioms:
  - **Closure**
    For all a and b in G, $a \circ b$ is also in G
  - **Associativity**
    For all a, b and c in G,
    $$(a \circ b) \circ c = a \circ (b \circ c)$$
  - **Existence of an identity element**
    There exists an element e in G such that, for all g in G,
    $$e \circ g = g \circ e = g$$
  - **Existence of an inverse element**
    For each a in G, there is an element b in G such that
    $$\mathrm{a} \circ b = b \circ a = e$$

Unicity, commutativity, ways of notation



- The associativity allows for the brackets to be dropped when applying the operation repeatedly, but the order remains
- **Exercise:**
  Prove that the identity element and the inverse of a given element from the group are unique!
- **Abelian (commutative) group**
  Thus is called a group having the property that for all f și g in G,
  $$f \circ g = g \circ f$$
- **Additive/multiplicative notation**
  Often in practice the group law $\circ$, the identity e, the inverse element of g and $\underbrace{g \circ g \circ \cdots \circ g}_{n}$ are denoted:
  - **additively** $(+,\ 0,\ -g,\ \mathrm{n}g)$,
  - **multiplicatively** $(\cdot,\ 1,\ g^{-1},\ g^{n})$

Group order; examples

- **A finite group**
  is a group having a finite number of elements, as opposed to an infinite group
- **The order of the group G, denoted by $|G|$**
  is the number of elements in the group, if finite, or ∞
- **Simple examples (exercise!)**
  Which of the following are groups? Which are abelian groups?
  - $(\mathbb{Z},+)$, $(\mathbb{Q},+)$, $(\mathbb{R},+)$, $(\mathbb{C},+)$
  - the additive group of a vector space
  - $(\mathbb{Z}^*,\cdot)$, $(\mathbb{Q}^*,\cdot)$, $(\mathbb{R}^*,\cdot)$
  - $(\mathbb{N},+)$, $(\mathbb{Z},\cdot)$
  - $(\mathbb{Z}_n,+)$
  - $GL_n(\mathrm{K})$, the set of all invertible matrices with elements taken from the field K, together with matrix multiplication

Homomorphism, Isomorphism

- **A group homomorphism**
  is a map $f: G_1 \to G_2$ having the property that, for all $a, b \in G_1$,
  $$f(a)f(b) = f(ab)$$
- Do the following hold for a homomorphism?
  $$\mathrm{f}(e_1) = e_2$$
  $$\mathrm{f}(a^{-1}) = [f(a)]^{-1}$$
- **A group isomorphism**
  is an invertible homomorphism, that is one for which there exists
  $f^{-1}: G_2 \to G_1$ such that
  $$f \circ f^{-1} = id_{G_2}$$
  $$f^{-1} \circ f = id_{G_1}$$
- Show that a homomorphism is an isomorphism if and only if it is bijective
- **An endomorphism**
  is a homomorphism $f: G \to G$, where the domain and codomain coincide
  An automorphism
  is an isomorphism $f: G \to G$, where the domain and codomain coincide
- The isomorphism relation satisfies the properties of an equivalence relation:
  - reflexivity
  - symmetry
  - Transitivity

Subgroups

- **Subgroup**
  A subset H of the group G is called a subgroup if, for all $a, b \in \mathrm{H}$:
  $$ab \in \mathrm{H}$$
  $$a^{-1} \in \mathrm{H}$$
- Prove that he two previous conditions can be replaced by just one,
  $$ab^{-1} \in \mathrm{H}$$
- **Notation**
  $$H \leq G$$
- **Trivial examples** are the improper groups
  $$\{1\}, \mathrm{G}$$
- **Is the intersections of two subgroups a subgroup? What about their union?**
- **Show that a subgroup of G is the center of G,**
  formed by all the elements in G that commute with all the other elements
  $$Z(G) = \{a \in G | ab = ba, \forall b \in G\}$$

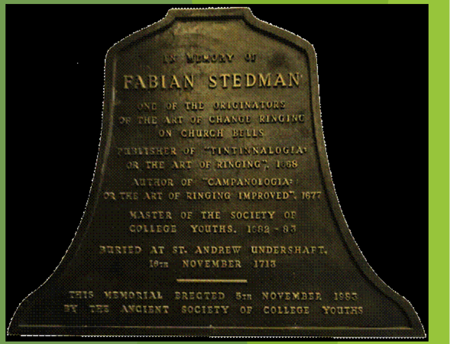
Subgroup generated by a set

- **Subgroup generated by a set $A \subseteq G$**
  We define $\langle A \rangle$ as the minimum subgroup containing A, i.e. the intersection of all subgroups of G that contain A.
  An element g of G is in $\langle A \rangle$ if and only if is a product of a finite number of either elements of A or their inverses
- **A finitely generated group**
  is a group $\mathrm{G} = \langle A \rangle$, where A is a finite set
- **A cyclic group**
  is a group generated by just one element
  $$\mathrm{G} = \langle a \rangle$$
- **Show that any cyclic group is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_n$**, where $n \in \mathbb{N}$!
- Which of $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Q}$ are cyclic or of finite type?

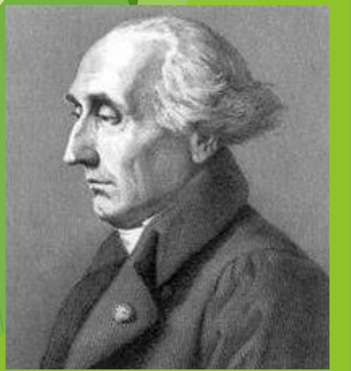
The order of an element in a group

- **The order of an element** a
in a group G is defined as the order of the subgroup it generates
$$\operatorname{ord} a = |\langle a \rangle|$$
- For a finite order element a of the group G the following hold:
  - $\operatorname{ord} a = \min\{n \in \mathbb{N}^* | a^n = 1\}$
  - $\operatorname{ord} a = n$ if and only if $a^n = 1$ and $1, a, a^2, \dots, a^{n-1}$ are all distinct
- A finite group only has finite order elements.
- There are infinite groups whose every element has a finite order, called periodic groups, such as $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$, ad infinitum.
- There are infinite groups whose only element of finite order is the identity, called torsion free groups, such as $\mathbb{Z}$
- If $f: G \to H$ is a homomorphism, and a is an element of G of finite order, then $ord\big(f(a)\big)$ divides $ord(a)$, being equal to it if f is injective. (in general the order of a subgroup divides the order of the group, as we shall see)
- Show that $ord(ab)$=$ord(ba)$. It is otherwise unrelated to $ord(a)$ and $ord(b)$

Transformation groups, examples



- **A transformation group**
  of a set A is a collection G of bijective (one to one and onto) transformations of the set A, having the properties:
  $$f_1, f_2 \in G \Rightarrow f_1 f_2 \in G$$
  $$f \in G \Rightarrow f^{-1} \in G$$
  $$\text{the identity } id_A \in G$$
- **G is a subgroup of $S(A)$,**
  the set of **all** bijective transformations of the set A
  $$S(A) = \{\mathrm{f: A} \rightarrow A | f \ bijective\}$$
- **A permutation group**
  is a transformation group of the set {1,2,…,n}, where $n \in \mathbb{N}^*$
- **The symmetric group S_n**
  is the set of **all** bijective transformations of the set $\{1,2,\ldots n\}$, $n \in \mathbb{N}^*$
  $$S_n = S(\{1,2,\ldots n\})$$
- We write any of the $|S| = \mathrm{n}!$ permutations in the form
  $$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}$$
- Show that S_n is non-abelian if n>2 !



Transformation groups, examples

- **Exercise!**
  Express all the elements in S_3 in terms of the identity e, the transposition $\tau = (1,2)$ and the cycle $\sigma = (1,3,2)$. Write the multiplication table for this group
- **General linear group**
  Let V be a vector space over a field K. From the set of all endomorphisms of V, $End_K(V) = \{f: V \to V | f \text{ linear map}\}$ we select the bijective ones (automorphisms), defining
  $$GL(V) = \{f \in End_K(V) | f \text{ bijective}\}$$
- **Prove the isomorphisms**
  - $End_K(V) \cong M_n(\mathrm{K})$ (with addition, and with multiplication, too, as ring isomorphism)
  - $GL(V) \cong GL_n(\mathrm{K})$ (with multiplication)
- **Prove that the translations in a vector space V**
  $$t_u: V \to V, t_u v \equiv u + v, \forall v \in V$$
  form a transformation group isomorphic to V, $T(V) = \{t_u | u \in V\}$

Unitary and Orthogonal Transformations

- **The Unitary Group**
  Given a complex vector space V, show that $U(V) \leq GL(V)$, where
  $$U(V) = \{f \in End_{\mathbb{C}}(V) | (f(u), f(v)) = (u, v), \forall u, v \in V\}$$
- **The Orthogonal Group**
  Given a real vector space V, show that $O(V) \leq GL(V)$, where
  $$O(V) = \{f \in End_{\mathbb{R}}(V) | (f(u), f(v)) = (u, v), \forall u, v \in V\}$$
- **Show that the set of unitary matrices**
  $$U(n) = \{A \in \mathcal{M}_n(\mathbb{C}) | A^* A = I_n\}$$
  is a subgroup of $GL_n(\mathbb{C})$ and that it is isomorphic to $U(V)$ if V is a complex vector space of dimension n
- **Show that the set of orthogonal matrices**
  $$O(n) = \{A \in \mathcal{M}_n(\mathbb{R}) | A^\intercal A = I_n\}$$
  is a subgroup of $GL_n(\mathbb{R})$ and that it is isomorphic to $O(V)$ if V is a real vector space of dimension n
- **Show that the set of unitary matrices with unit determinant**
  $$SU(n) = \{A \in U(n) | \det A = 1\}$$
  is a subgroup of $U(n)$
- **Show that the set of orthogonal matrices with unit determinant**
  $$SO(n) = \{A \in O(n) | \det A = 1\}$$
  is a subgroup of $U(n)$
- **How many independent real/complex parameters**
  are needed to describe the groups $O(n)$ **and** $U(n)$, respectively? What about $SO(n)$ **and** $SU(n)$?

Some exercises

- **Study the dihedral group** D_n
  of the isometries of the regular polygon with n sides. Show that
  $$D_3 \cong S_3$$
- What are the subgroups of $D_3 \cong S_3$? What is the centre of this group?
- Prove the **Rearrangement Theorem**,
  For every element $f \in G$, the sets $\{fg|g \in G\}$ and $\{gf|g \in G\}$ contain every element once and only once.

Equivalence relations

- **A binary relation on a set A**
  is a collection of ordered pairs of elements of A, i.e. a subset of the Cartesian product $A^2 = A \times A$.
- For an **equivalence relation**,
  remember the three necessary conditions:
  - Reflexivity
  - Symmetry
  - Transitivity
- **The equivalence class of x**
  If "~" is an equivalence relation on the set M we define the above as:
  $$C_x = \{y \in M | y \sim x\}$$
- **Partition of M**
  Prove that an equivalence relation divides a set M into a set of disjoint equivalence classes whose reunion is the set M
  $$x, y \in M \Rightarrow C_x \cap C_y = \emptyset \text{ or } C_x = C_y$$
  $$M = \bigcup_{x \in M} C_x$$
- **The quotient set $\boldsymbol{M/\sim}$**
  is the set of all equivalence classes,
  $$M/\sim = \{C_x | x \in M\}$$

Cosets

- **Left and right cosets of a subgroup $\boldsymbol{H} \leq G$ with respect to an element $\boldsymbol{g}$**

$$gH = \{gh|h \in H\}$$
$$Hg = \{hg|h \in H\}$$

- **Prove that the above can be defined as partition classes of G** introduced by equivalence relations defined as:

$$x\sim_l y \text{ iff } x^{-1}y \in H$$
$$x\sim_r y \text{ iff } yx^{-1} \in H$$

- Do these classes form subgroups?
- Find a well-defined, **bijective mapping** between the quotient sets,

$$\mathrm{f}: \mathrm{G}/\sim_l \to \mathrm{G}/\sim_r$$

- **The index of a subgroup $\boldsymbol{H}$ in a group $\boldsymbol{G}, |\boldsymbol{G}:\boldsymbol{H}|$** is defined as the number of elements of any of the above

$$|G:H| = |\mathrm{G}/\sim_l| = |\mathrm{G}\,/\sim_r|$$

- Study the cosets for $D_3 \cong S_3$

Lagrange's Theorem

- Prove **Lagrange's Theorem**
  $$|G| = |H||G{:}\,H|$$
  (prove all classes have a given number of elements)
- Show that the order of an element divides the order of the group
- Classification of some low order groups
  Show that:
- n prime $\Leftrightarrow$ G is cyclic, isomorphic to $\mathbb{Z}_p$
- n=4 $\Leftrightarrow G \cong \mathbb{Z}_4$ or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (the Klein four-group)
- n=6 $\Leftrightarrow G \cong \mathbb{Z}_6$ or $G \cong S_3$
- n=8, $\text{non}-\text{abelian} \Leftrightarrow G \cong D_4$ or $G \cong Q$ (group of quaternions)
- n=8, $\text{abelian} \Leftrightarrow G \cong \mathbb{Z}_8,\ G \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Conjugacy classes

- **Two elements $\boldsymbol{x, y \in G}$ are said to be conjugate**
  if there exists an element $g \in G$ such that $x = gyg^{-1}$
- **Show the above relation is an equivalence one!**
- **The conjugacy class of an element a**
  is defined as:
  $$(a) = \{b \in G | a \sim b\}$$
- **Find the conjugacy classes of**
  - An abelian group
  - The dihedral group D_3
- Argue that in general, the number of conjugacy classes in the symmetric group S_n is equal to the number of integer partitions of n

Normal groups

- **A subgroup $\boldsymbol{H \leq G}$ is called normal (or self-conjugate)** if, $\forall h \in H, g \in G \Rightarrow\ ghg^{-1} \in H$
- Notation: $\boldsymbol{H \trianglelefteq G}$
- **An inner automorphism of $\boldsymbol{G}$**
$$\varphi_g : \mathrm{G} \to G, \varphi_g(x) = gxg^{-1}$$
- **$\boldsymbol{H \leq G}$ is normal iff it is invariant to any inner automorphism of $\boldsymbol{G}$**
- **Show that the following are normal subgroups:**
  - $\{1\}, G$
  - The kernel of a group homomorphism $f : \mathrm{G} \to G'$
$$\mathrm{Ker\ f} = \{x \in G | f(x) = e_{G'}\}$$
  - Any subgroup of an abelian group
  - Any subgroup of index 2 in a group G
- Notice that the image of a subgroup through a group homomorphism $f : G \to G'$ is a subgroup but if $H \trianglelefteq G$ we are not sure $\mathrm{f}(H) \trianglelefteq G'$ Provide a counterexample!

Correspondence of (normal) subgroups

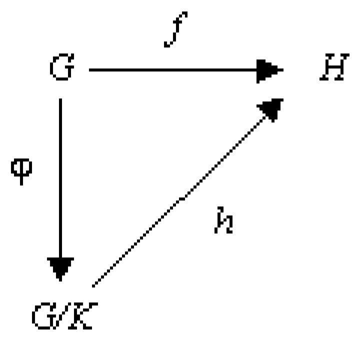
- Prove that for any group homomorphism $f\colon \mathrm{G} \to G'$ (not necessarily bijective)
  - $H' \trianglelefteq G' \Rightarrow f^{-1}(H') \trianglelefteq G$
  - If f is surjective then $H \trianglelefteq G$ implies $\mathrm{f}(H) \trianglelefteq G'$
  - If f is surjective then there is a bijective correspondence between the set of all subgroups of G that contain $\mathrm{Ker\,f}$ and the set of all subgroups of G'
  - The same as the above but for subgroups $\rightarrow$ normal subgroups
- Study the subgroups of $\mathbb{Z}$. Show that:
  - $n\mathbb{Z} \leq \mathbb{Z}$, where $n\mathbb{Z} = \{nz | z \in \mathbb{Z}\}$
  - If $H \leq \mathbb{Z}$, then there exists an $\mathrm{n} \in \mathbb{N}$ such that $\mathrm{H} = n\mathbb{Z}$
  - $n\mathbb{Z} \leq m\mathbb{Z} \Leftrightarrow m$ divides n
- Determine the subgroups of $\mathbb{Z}_n$
  using the correspondence with the subgroups of $\mathbb{Z}$ that contain $\mathrm{ker} f$
- A group is called simple if $\mathrm{G} \neq \{1\}$ and its only normal subgroups are $\{1\}$ and G
- The only abelian simple groups are the cyclic groups of prime order!

Quotient group

- Show that, if $\boldsymbol{H} \trianglelefteq \boldsymbol{G}$ is a normal subgroup, then
$$\mathrm{G}/\sim_l = \mathrm{G}/\sim_r \equiv G/H$$
- Also show that the above forms a group together with the operation
$$(xH)(yH) = xyH$$
- Prove that a surjective homomorphism $p: \mathrm{G} \to G/H$ is the **canonical surjection,**
$$p(x) = xH$$
- Determine the kernel of $p(x)$!
- What is the order of G/H?
- Prove the first isomorphism theorem
  - The kernel of a homomorphism $f: G \to H$ is a normal subgroup of G
  - The image of a homomorphism $f: G \to H$ is a subgroup of H
  - $\mathrm{Im} f \cong G/\mathrm{ker} f$

Universality of the factor group

- **Universality of the factor group**
  Show that if $K \trianglelefteq \boldsymbol{G}$, $\varphi: \mathrm{G} \to G/K$ is the canonical surjection, and $f: \mathrm{G} \to H$ is a group homomorphism
  - $\exists h: G/K \to H$ group homomorphism such that $h \circ \varphi = f \Leftrightarrow \ker\varphi \leq \ker f$
    If it exists, it is unique!
  - If h exists, then it is surjective iff f is surjective
  - If h exists and is injective, then $\ker \varphi = \ker f$



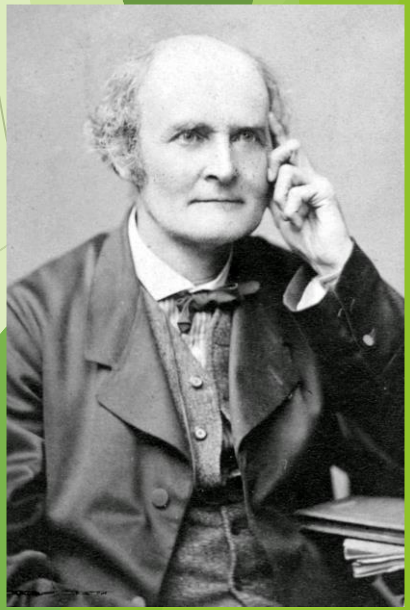
- **Fundamental theorem on homomorphisms**
  Let $K \trianglelefteq \boldsymbol{G}$, $\varphi: \mathrm{G} \to G/K$ is the canonical surjection, and $f: \mathrm{G} \to H$ is a group homomorphism.
  If $\mathrm{K} \leq \ker f$ then ∃! a homomorphism h:G/K→H such that f = h φ.
- If $\mathrm{K} = \ker f$ then ∃! h:G/Ker f→Im f isomorphism, called the canonical one

Exercises

- If G is a group, $Z(G)$ is its center, and $\Psi\colon \mathrm{G} \to \mathrm{Aut}(G)$ with
$$\Psi(g) = \varphi_g, \text{where } \varphi_g\colon \mathrm{G} \to G, \varphi_g(x) = gxg^{-1}$$
(inner automorphisms, forming the set $\mathrm{Inn}(G)$)
show that $G/Z(G) \cong \mathrm{Inn}(G)$.
(Show first that Ψ is a group homomorphism, that $\mathrm{Im}\Psi \equiv \mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$, that $\mathrm{Ker}\Psi = Z(G)$)
- Show that
$$\mathbb{Z}/(n\mathbb{Z}) \cong \mathbb{Z}_n.$$
- Show that the signature of a permutation,
$$\varepsilon(\sigma) = \prod_{i<j} \frac{\sigma(j) - \sigma(i)}{j - i}$$
is a morphism from S_n to {-1,1} with multiplication as group law.
- Show that
$$\mathrm{Ker}\ \varepsilon \equiv A_n \trianglelefteq S_n$$
$$S_n/A_n \cong \{-1,1\}$$
$$|A_n| = \frac{n!}{2}$$
- What is the order of a cycle? How can a permutation be decomposed uniquely into cycles?

Cayley's Theorem

- **Theorem**
  Any permutation can be decomposed uniquely (up to permutations of the factors) into a product of disjoint cycles!
  Do this for:
  $$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 9 & 5 & 8 & 7 & 4 & 1 & 6 & 2 \end{pmatrix}$$
- **Theorem**
  Any group with n elements is isomorphic to a subgroup of the permutation subgroup S_n
  Proof:
  Define $u_g\colon \mathrm{G} \to G,\ u_g(x) = gx$
  Prove that:
  $u_g u_h = u_{gh}$, so $u_g u_{g^{-1}} = id_G$ and therefore $u_g \in S(G)$
  $\Psi\colon \mathrm{G} \to \mathrm{S}(G), \Psi(g) = u_g$ is a group homomorphism
  $\mathrm{Ker}\Psi = \{1\}$
  Therefore $\mathrm{Im}\Psi \cong G$
- **Definition:** A group G is called simple iff $\mathrm{G} \neq \{1\}$ and it has no normal groups apart from $\{1\}$ and G
- Prove that an abelian group G is simple iff it is isomorphic to $\mathbb{Z}_p$, where p is a prime number.



Direct and semidirect products

- Direct product
  If G_1, G_2 are groups, $G_1 \times G_2$ forms a group together with the group law
  $$(g_1, g_2)(h_1, h_2) \equiv (g_1 h_1, g_2 h_2)$$
- Theorem 1 about $G_1 \times G_2$
  i) it contains a subgroup isomorphic to G_1, formed by the elements (g_1, e_2)
  ii) it contains a subgroup isomorphic to G_2, formed by the elements (e_1, g_2)
  iii) the elements of these two subgroups commute with each other
  iv) the only common element of the two subgroups is (e_1, e_2)
  v) any element of the group is the product of two elements of the two subgroups
- Extending the above definition, any group that is isomorphic to one constructed as above is called a direct product group
- Theorem 2
  If a group G contains two subgroups, G'_1 and G'_2, such that:
  i) any two ellements from different subgroups commute with each other,
  ii) G_1' and G_2' only have the identity in common,
  iii) any element of the group is the product of two elements of the two subgroups,
  then G is a direct product group

Direct and semidirect products

- Prove that O(3) is a direct product group
- Prove that the first condition in theorem 2 is equivalent to the following: ${G'}_1$and ${G'}_2$ are both normal subgroups of G
- To define a semi-direct group we weaken this condition by allowing ${G'}_2$ to be any kind of subgroup
- Definition: A group G is called a semi-direct product group if it possesses two subgroups, ${G'}_1$ and ${G'}_2$, such that:
  i) ${G'}_1 \trianglelefteq G$
  ii) $G_1{}'$ and $G_2{}'$ only have the identity in common,
  iii) any element of the group is the product of two elements of the two subgroups
- For both direct and semi-direct products, ii) implies that the decomposition iii) is unique

The Euclidean group of $\mathbb{R}^3$

- The group of all linear transformation of the Euclidean space, has elements of the form $T = \{R(T), \boldsymbol{t}(T)\}$, where R describes a rotation and **t** a translation so that:

$$\mathbf{r}' = T\boldsymbol{r} = \mathrm{R(T)}\mathbf{r} + \boldsymbol{t}(T)$$

- Show that indeed this forms a group, writing the explicit form of the group law and of the inverse.
- We refer to the transformations with $\boldsymbol{t}$=**0** as pure rotations and to those with R=I as pure translations
- Prove that this group is a semi-direct product group

Group representations

- **A linear representation of a group G**
  onto a vector space V, having the scalar field K (real/complex numbers in our applications) is a group homomorphism
  $$D: G \to GL(V)$$
  The dimension of the representation is said to be the dimension of V
- **A matrix representation of a group G**
  onto a vector space V, having the scalar field K (real/complex numbers in our applications) is a group homomorphism
  $$D: G \to GL_n(\mathrm{K})$$
- Real/complex representations
- Find a linear representation for Klein's group and for the cyclic group of order 3
- Construct matrix representations starting from a linear representation of a group G
- Two n-dimensional matrix representations $D_1: G \to GL_n(\mathrm{K})$, $D_2: G \to GL_n(\mathrm{K})$ are said **equivalent** if there exists $S \in GL_n(\mathrm{K})$ such that
  $$D_2(g) = SD_1(g)S^{-1}, \forall g \in G$$

Further examples

- $R: \mathbb{R} \to GL(\mathbb{R}^2)$ given by the rotation with angle $\alpha \to R(\alpha)$ is a two-dimensional real representation of the group $(\mathbb{R}, +)$
- The trivial representation, $\mathrm{D}: G \to \mathbb{C}^*$, $D(g) = 1$
- For a group of matrices, $\mathrm{D}: G \to \mathbb{C}^*$, $D(g) = \det(g)$ provides a non-trivial 1D representation

The characters, invariant spaces, irreducible representations

- The character of a representation D is the set
$$\chi = \{\chi(\mathrm{g}) | \mathrm{g} \in G\}, \text{where}$$
$$\chi(\mathrm{g}) = \mathrm{Tr}[\mathrm{D}(\mathrm{g})]$$
- The characters are functions of equivalence classes
- Definition: Let $D: G \to GL(V)$ be a linear representation. A subspace W of V is called **invariant** with respect to D if $\mathrm{D}(\mathrm{g})W \subseteq W, \forall g \in G$
- Definition: A linear representation $D: G \to GL(V)$ is said to be irreducible if its only invariant subspaces are the trivial ones, $\{0\}$ and V
- Definition: A linear representation $D: G \to GL(V)$ is said to be reducible if it is not irreducible.
- Prove that $R: \mathbb{R} \to GL(\mathbb{R}^3)$, given by the rotation matrix with angle $\alpha \to R(\alpha)$ around the Oz axis, is a reducible representation of the group $(\mathbb{R}, +)$
- Definition: Sum of representations
If $D_1: G \to GL(V_1)$, $D_2: G \to GL(V_2)$ are linear representations, show that we can define a linear representation $D: G \to GL(V_1 \oplus V_2)$, by
$$D(g)(x_1, x_2) = (D_1(g)x_1, D_2(g)x_2), \forall (x_1, x_2) \in V_1 \oplus V_2$$

Reducible representations

- Definition: A linear representation $D: G \rightarrow GL(V)$ is said to be completely reducible if for any invariant space $U \leq V$ there exists a complement invariant subspace $W \leq V$ such that $V = U \oplus W$
- Describe the one-dimensional representations of a group in terms of reductibility.
- Two linear representations of a group, $D_1: G \rightarrow GL(V)$ and $D_2: G \rightarrow GL(V)$ are called equivalent if there exists a linear isomorphism $S: G \rightarrow GL(V)$, such that, for any $g \in G$
$$SD_1(g) = D_2(g)S$$
- Prove that if two linear representations of a group, $D_1: G \rightarrow GL(V)$ and $D_2: G \rightarrow GL(V)$ are equivalent and D_1 is irreducible, the same holds for D_2, too.
- Prove that if $D: G \rightarrow GL(V)$ is completely reducible, it can be written as a direct sum of irreducible representations.

Unitary/Orthogonal representations

- **A unitary representation of a group G**
  onto a complex vector space V is a group homomorphism
  $$D: G \to U(V)$$
- Prove that any unitary representation is a completely reducible one.
- **An orthogonal representation of a group G**
  onto a real vector space V is a group homomorphism
  $$D: G \to O(V)$$
- Similarly define unitary/orthogonal matrix representations