Representations of finite groups

Theorem (Heinrich Maschke):
Let G be a finite group over $K = \mathbb{R}$ or $\mathbb{C}$. Then any finite representation of G is completely reducible.
Proof: Let $D : G \to GL\left(V\right)$, $U \leq V$ invariant subspace. We have X subspace such as $U \oplus X = V$, but need to prove it is invariant.
We use the projection $\pi : V \to U$ and define

$$\pi'\left(x\right) = \frac{1}{\left|G\right|} \sum_{g\in G} D\left(g\right) \pi \left(D\left(g\right)^{-1}\left(x\right)\right)$$

π' is linear and $\pi'\left(V\right) = U$, $\pi' : V \overset{D(g)^{-1}}{\to} V \overset{\pi}{\to} U \overset{D(g)}{\to} U$
Show that $\operatorname{Im} \pi' = U$.
Show that $U \oplus \ker \pi' = V$.
Now show that $\ker \pi'$ is invariant.
Corrolary: Any representation of a finite group is a direct sum of irreducible representations

Lie Groups

Lie groups can be defined in general in several ways, as topological groups with additional analytic properties, or as an analytic manifold to which group properties are added. The general definition can be rather abstract and involved, but the cases of physical interest are belonging to a special type, a linear Lie group, which is straightforward to define

Definition: A linear Lie group of dimension

A group G is a linear Lie group of dimension n if it satisfies the following conditions:

1. G must posess at least one faithful finite dimensional representation Γ

if it has dimension m then we can define a metric, by

$$d\left(g, g'\right) = \sqrt{\sum_{i,j=1}^{m} \left|\Gamma\left(g\right)_{ij} - \Gamma\left(g'\right)_{\acute{\imath}j}\right|^{2}}$$

Lie Groups

We thus endow it with the topology of the complex Euclidean space $\mathbb{C}^{m^2}$
Properties of a metric:
i) $d\left(g, g'\right) = d\left(g', g\right)$
ii) $d\left(g, g\right) = 0$
iii) $d\left(g, g'\right) > 0$ if $g \neq g'$
iv) $d\left(g, g'\right) + d\left(g', g''\right) > d\left(g, g''\right)$
Let M_δ be a spherical vicinity of the identity,

$$M_\delta = \left\{g \middle| d\left(g, e\right) < \delta\right\}$$

Lie Groups

2. There exists a δ such that any point in the sphere of radius δ, M_δ can be parametrised **uniquely** by n real parameters $x_1, x_2, \ldots, x_n$. The identity corresponds to $x_1 = x_2 = \ldots = x_n = 0$.
3. There exists a radius η such that any point in $\mathbb{R}^n$ belonging to the sphere

$$R_\eta = \left\{ \mathbf{x} \middle| \sum_{j=1}^{n} x_j^2 < \eta^2 \right\}$$

corresponds to some element in M_δ. So there is a one-to-one correspondence.
4. Each of the matrix elements $\Gamma(x_1, x_{2,} \ldots, x_n)$ must be an analytic function of $x_1, x_{2,} \ldots, x_n$, for any $\mathbf{x} = (x_1, x_{2,} \ldots, x_n) \in R_\eta$

Lie Groups

Let us define the n $m \times m$ matrices $\mathbf{a}_p$:

$$(a_p)_{jk} = \left(\frac{\partial \Gamma_{jk}}{\partial x_p} \right)_{\mathbf{x}=\mathbf{0}}$$

Theorem: The matrices $\mathbf{a}_1$, ..., $\mathbf{a}_n$ form the basis of a real n dimensional vector space.

Prove it, using the fact that, from the set

$$S' = \{\operatorname{Re}\Gamma_{11}, \operatorname{Re}\Gamma_{12}, ..., \operatorname{Re}\Gamma_{mm}, \operatorname{Im}\Gamma_{11}, \operatorname{Im}\Gamma_{12}, ..., \operatorname{Im}\Gamma_{mm}\}$$

one can always choose a subset S with n members, such that the rest are analytic functions of the ones in S

Notice that $\mathbf{a}_p$ are, however, not necessarily real matrices!

We'll show later they form the basis of a Lie algebra.

Examples

1) Prove that $(\mathbb{R}^*, \cdot)$ is a linear Lie group
2) Do the same for $O(2)$, $SO(2)$
Hint:$\delta = \sqrt{2}$ allows us to include only proper rotations in the parametrisation
3) Do the same for $SU(2)$

$$u = \begin{pmatrix} p_0 + ip_3 & p_2 + ip_1 \\ -p_2 + ip_1 & p_0 - ip_3 \end{pmatrix}$$

Find the generators of the Lie algebra a_p
4) The Euclidean group of $\mathbb{R}^3$

Connected components of a linear Lie group

Definition: *A connected component of a linear Lie group* is a maximal set of elements $g \in G$ that can be obtained by continuously varying one or more matrix elements $\Gamma_{jk}(g)$ of the faithful representation Γ.

Apply to the previously studied examples (1&2)!

Prove the following

Theorem: The connected component of a linear group G containing the identity is an invariant subgroup of G, called the connected subgroup. Every connected component of G is a coset of the connected subgroup.

- we may have a countably infinite number of connected components, but not in physically interesting cases

Definition: *A linear Lie group is called connected* if it posesses only one connected component

A connected Lie group can be parametrised by a set of n real parameters $(y_1, y_2, ..., y_n)$, forming a connected set in $\mathbb{R}^n$, such that $\Gamma_{jk}(g)$ are continuous functions of these parameters. (not necessarily analytic, one-to-one)

Examples!

Compactness

Theorem: A subset of a finite dimensional (real or complex) Euclidean space is compact if and only if it is closed and bounded.

A linear Lie group is compact if the parameters $y_1, ...y_n$ range over closed intervals, $a_j \leq y_j \leq b_j$

The physical non-compact Lie groups we encounter have unbounded matrix elements $\Gamma_{jk}(g)$ so, in practice, we identify the compact groups by the condition $d(g, e) < M$, $\forall g \in G$

A non-compact Lie group may have compact Lie subgroups.

Are $(\mathbb{R}^*, \cdot)$, $O(2)$, $SO(2)$, $U(2)$, $SU(2)$ compact?

Hurwitz integral on a compact Lie group

Let G be a compact Lie group parametrised by
$g \to \gamma = \{\gamma_1, \gamma_2, ...\gamma_n\} \in \Gamma \subset \mathbb{R}^n$
Let $f : G \to \mathbb{C}$, we may use the notation $f\left(\gamma\left(g\right)\right) \equiv f\left(g\right)$
We define the following integral:

$$\int d\mu\left(g\right) f\left(g\right) \equiv N_\gamma \int_\Gamma d\gamma\left(g\right) \left|\frac{D\gamma\left(g'\right)}{D\gamma\left(gg'\right)}\right|_{g'=e} f\left(\gamma\left(g\right)\right)$$

where $d\gamma\left(g\right) = d\gamma_1 d\gamma_2 ... d\gamma_n$
What should be the normalisation factor such that:

$$\int d\mu\left(g\right) = 1$$

Property 1:Show that if we choose a different parametrisation $g \to \delta = \{\delta_1, \delta_2, ...\delta_n\} \in \Delta \subset \mathbb{R}^n$ such that $\frac{D\delta(g)}{D\gamma(g)} \neq 0$, we get the same result, so the definition is parametrisation-independent.

Property 2: Let $F : G \to G$ be a bijective function $(g \leftrightarrow \tilde{g})$. Show that

$$\int d\mu\left(g\right) f\left(\tilde{g}\right) = \int d\mu\left(g\right) f\left(g\right)$$

Examples: $\tilde{g} = g^{-1}$, $\tilde{g} = xg$, $\tilde{g} = gx$, $\tilde{g} = xgx^{-1}, \ldots$

Write the Hurwitz integral on SU(2) in the Euler-Rodrigues

parametrisation!

$$\int d\mu\left(U\right) f\left(U\right) \equiv \frac{1}{2\pi^2} \int_{|\vec{p}|<1} d^3p \frac{f\left(-\left|p_0\right|, \vec{p}\right) + f\left(\left|p_0\right|, \vec{p}\right)}{\left|p_0\right|}$$

$$\int d\mu\left(U\right) f\left(U\right) \equiv \frac{2}{2\pi^2} \int_{|\vec{p}|<1} d^4p f\left(p\right) \delta\left(p^2 - 1\right) = \frac{2}{2\pi^2} \int_{S^3} d^3p f\left(p\right)$$

Other parametrisations

$$\int d\mu\left(U\right)f\left(U\right)\equiv\frac{1}{8\pi^2}\int_0^{2\pi}d\alpha\left(1-\cos\alpha\right)\int d\Omega_{\vec{u}}\left[f\left(\vec{u},\alpha\right)+f\left(\vec{u},\pi-\alpha\right)\right]$$

$$\int d\mu\left(U\right)f\left(U\right)\equiv\frac{1}{8\pi^2}\int_{\left|\vec{\xi}\right|\leq2\pi}d^3\xi\frac{1-\cos\xi}{\xi^2}f\left(\xi\right)$$

group

$SU(2)$

A function in $SU(2)$ is called even/odd if $F(-U) = \pm F(U)$

Show the following!

Property 1: Any representation of the group $SO(3)$ can be thought as an even representation of the group $SU(2)$ and vice-versa.

Property 2: $D(U) = \pm D(-U) \Leftrightarrow D(I) = \pm D(-I)$

Show the following

Theorem: A continuous representation of $SU\left(2\right)$ on a finite dimension Hilbert space is equivalent to a unitary representation on the same space

Hints: Define a new inner product

$$\left\langle u, v\right\rangle = \int d\mu\left(U\right)\left(D\left(U\right)u, D\left(U\right)v\right)$$

Between two orthonormal bases $\left(e'_j, e'_k\right) = \delta_{jk}$ and $\left\langle e'_j, e'_k\right\rangle = \delta_{jk}$ there is a bijective linear mappring $e'_j = Ae_j$

$\left(Au, Av\right) = \left\langle u, v\right\rangle$

$D'\left(U\right) = AD\left(u\right)A^{-1}$

Studying finite continuous representations on a compact Lie group is reduced to the study of unitary representations for which we know they are completely reducible. This result can be extended to representations on Hilbert spaces with countable bases.

For an analytic representation of a group, let us define the infinitesimal generators:

$$-iJ_k \equiv \left.\frac{\partial D\left(\vec{\xi}\right)}{\partial \xi_k}\right|_{\vec{\xi}=\vec{0}}$$

$J_k : V \to V$ linear operator

Notice that $\left.\frac{\partial D\left(\lambda\vec{\xi}\right)}{\partial\lambda}\right|_{\lambda=0} = -i\xi_k J_k$. Show that:

Theorem 1

Any operator from the representation can be written in exponential form:

$$D\left(\vec{\xi}\right) = \exp\left(-i\xi_k J_k\right)$$

Theorem 2

The representation D described above is unitary if and only if its infinitesimal generators are hermitic operators.

Theorem 3

Let $D\left(U\right)$ be an irreducible unitary representation of $SU\left(2\right)$
Let us consider the equation $J_3 f_m = m f_m$
Define $J_\pm = J_x \pm i J_y$. We have that:

$$[J_3, J_\pm] = \pm J_\pm, \;\; [J_+, J_-] = 2J_3$$
$$J_3(J_\pm f_m) = (m \pm 1) J_+ f_m$$

It follows that $J_+ f_m = \beta_m f_{m+1}$, $J_- f_m = \alpha_m f_{m-1}$, or zero.
Due to irreducibility, the eigenvalues are nondegenerate.
Show that $\beta_m = \alpha^*_{m+1}$, if the eigenvectors are normalized to unity.
Prove the finite difference equation:

$$\left|\alpha_m\right|^2 - \left|\alpha_{m+1}\right|^2 = 2m$$

Prove that $|\alpha_m|^2 = -m^2 + m + c$

Show that, since m must be bounded, $j' \leq m \leq j$, $j - j' = N - 1$, N positive integer:

$c = j\,(j+1)$, $j' = -j$, hence $2j = N - 1$

Therefore $j \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\right\}$

$\alpha_{j+1} = \alpha_{-j} = 0$

$\dim V^{(j)} = N = 2j + 1$

Why cannot we have a second set of vectors f'_m orthogonal to f_m?

We choose the phases of the vectors such as $\alpha_m = \sqrt{(j+m)\,(j-m+1)}$

Any irreducible $SU(2)$ representation is finite dimensional
$V^{(j)} = Span\{f_{jm}\},\ f_{jm} \equiv f_m$, canonical basis
Show that $J^2 = j(j+1)I$
$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$
Show that:
$D^{(j)}_{mn}(U(\vec{e}_3, \alpha)) = e^{-im\alpha}\delta_{nm}$
the representations with integer maximum weight j are even, the other are odd
$D^{(j)}(-U) = (-1)^{2j} D^{(j)}(U)$
those with integer maximum weight are irreps of $SO(3)$, the other are not

The character is a function of the equivalence class. Show that:

$$\chi_j\left(U\right) = \frac{\sin\left(2j+1\right)\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}$$

Show that:

$$D^{(j)}_{nm}\left(\phi,\theta,\psi\right) = e^{-im\phi} d^{(j)}_{mn}\left(\theta\right) e^{-in\psi}$$

$$d^{(j)}_{mn} = \left(f_{jm}, e^{-i\theta J_2} f_{jn}\right)$$

Schur Lemmas

Theorem 1: Let D and D' be two irreducible representations of a group G of dimensions $d,\ d'$ and let us assume there exists a $d \times d'$ matrix A such that:

$$D\left(g\right)A = AD\left(g\right)'$$

for all $g \in G$. Then one either has $A = 0$ or $d = d'$ and $\det A \neq 0$.

Theorem 2: Let D be an irreducible representation of a group G of dimension d and let us assume there exists a $d \times d$ matrix A such that:

$$D\left(g\right)A = AD\left(g\right)$$

for all $g \in G$. Then A is a multiple of the identity.

Prove that any irreducible representation of an Abelian group is one-dimensional

Show that
$H = L^2\left(SU\left(2\right)\right)$ is a Hilbert space if we define
$\left(f, g\right) = \int d\mu\left(U\right) f^*\left(U\right) g\left(U\right)$, dot product
Let $A \equiv \int d\mu\left(U\right) D^{(j)}\left(U\right) B D^{(j')}\left(U^{-1}\right)$
Show that $D^{(j)}\left(U_0\right) A = A\ D^{(j')}\left(U_0\right)$, for any $U_0 \in SU\left(2\right)$
Applying Schur's lemmas:
$j' \neq j \Rightarrow A = 0$
$j' = j \Rightarrow A = \lambda I_n$
Then prove that $TrB = TrA = \lambda N \Rightarrow \lambda = \frac{TrB}{2j+1}$
Show that $A_{mm'} = B_{nn'}\left(D^{(j')}_{m',n'}, D^{(j)}_{mn}\right)$

Choosing $B_{nm} = \delta_{np}\delta_{n'p'}$

$A_{mm'} = \left(D^{(j')}_{m'p'}, D^{(j)}_{mp}\right)$

Show that

$\left(D^{(j')}_{m'p'}, D^{(j)}_{mp}\right) = \frac{1}{2j+1}\delta_{j'j}\delta_{m'm}\delta_{n'n}$

$\left(\chi^{(j')}, \chi^{(j)}\right) = \delta_{jj'}$

$j = 0$, $D^{(0)}\left(U\right) = 1$

$j = 1/2$, $D^{(1/2)}\left(U\right) = U$

What happens for $j = 1$?